

welcome back students in the previous lectures we have been looking at applications of matrices especially on the usage of what is known as the row reduced echelon matrix in this lecture we will use this row reduce echelon matrix and try to solve a system of linear equations

so let us begin with this is the system of linear equation plus a one $n \times n$ equal to b one $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$ up to $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$

so this is a system of m equations with n unknowns not with in n unknowns right so what you what we have here is a system of m equations in n unknowns now given a system how to solve it before we proceed further to solving it here are some of the facts that one will have to observe

so in fact there are three possibilities that arise given a system what are they first one no solution a solution may not exist you may not be find solutions to $x_1 \times x_2 \times x_n$ satisfying the given system second unique solution and finally multiple solution in fact when you say that it is a multiple solution it is going to be an infinite solution infinite set of solution

so let us try to see how to use this row reduced echelon matrix to solve a given system of equation

so let us do an example the given system is minus three x minus two y plus four z equal to nine three y minus two z equal to five four x minus three y plus two z equal to seven given the system let us try to write down in the form of matrix minus three minus two four zero 3 minus 2 4 minus 3 2 and evaluated at $x \ y \ z$ what we get is nine five seven right

so these nine five seven they are called the constant terms of the equation and this matrix that we have here this is what is known as the coefficient matrix and this $x \ y \ z$ are the unknowns

so this is the coefficient matrix these are the unknowns and these are the constant terms now given the system right we have written down in terms of the matrices now let us try to write down in the form that we needed

so what we are going to do is write the coefficient matrix and then augment it with the constant of matrix right minus three minus two four zero three minus two four minus three two we are going to augment this with the constant matrix we are going to solve the system by using this augmented matrix

so first before we try to solve let us try to convert this into the let us try to convert the given coefficient matrix into the row reduced echelon matrix first thing that we will have to do is that find the leading coefficient or the first non zero coefficient

so the first non zero coefficient the first row is the is minus three

so lets make it into one

so what we are going to do is replace r_1 by one upon minus three times r_1

so what we will have is one two by three minus four by three apply the same set of operations on the constant of matrices also

so you will have minus three when you apply minus one by three you get it as minus three zero three minus two five four minus three two seven next thing is to make the other elements of that column into zero which means we have one here so so the one one of them is zero

so we should convert this four into zero r_3 will replace it by r_3 minus four times r_1 one zero zero

so we will have the same thing two by three minus four by three three minus two will have to

so we have made this into zero

so similarly multiply this one two by three by minus four

so you will have minus eight by three ah minus three minus eight by three and similarly two plus 16 by 3 you have minus 3 5 7 plus twelve and therefore the

resulting matrix that we have over here is one two by three minus four by three augmented with minus three zero three minus two five zero minus seventeen by three twenty two by three nineteen now let us look at this sub matrix that we have

so or the leaving the first column and the first row let us look at the remaining two by two sub matrix here again the leading coefficient here which is three its again non zero we will have to make them into one

so we will replace r_2 by one by three times r_2 one two by three minus four by three and then you will have minus three zero one minus two by three five by three zero minus seventeen by three twenty two by three and ninety now given this let us convert the other elements of that of this of the second column into zero r_1 is replaced by r_1 minus three by two times r_2 and similarly for the other one r_3 is replaced by r_3 plus three by seventeen times sorry it should be two by three ah two by three and here plus seventeen by three times r_2

so what we will have is one zero zero and here we are multiplying

so two by three minus two by three

so you have zero minus four by three plus four by nine and then here we are augmenting it with minus three minus ten by three and then we have one minus two by three five by three here it is going to be zero twenty two by three minus thirty four by nine and finally we have nineteen plus eighty five by nine right now let us write down the final matrix that we have what we have over here is one zero zero zero one zero we have minus four by three plus four by nine

so the resultant is going to be what we have is minus eight by nine minus two by three and similarly we have sixty six minus thirty four sixty six minus thirty four is going to be two three thirty two by nine and similarly for the other one minus 19 by 3 5 by 3 181 sorry 171 plus 85

so you have 6 right 171 plus 85 6 7 plus 8 it is going to be 15 1

so 256 upon nine and finally we are left with this one

so let us convert this into one

so we are going to replace r_3 by nine by thirty two times r_3 you have one zero minus eight by nine times eight by nine minus nineteen by three zero one minus two by three five by three zero zero one and then we will have two fifty six upon thirty two now now let us make the other two elements in the third column into zeros

so will replace r_1 by eight by nine times r_3 plus r_1 and similarly r_2 by two by three times r_3 plus r_2 what we will end up with here is just the identity matrix but if you notice on the other end what we will have is just 3 7 and 8.

and

so now let us try to write down the final one in terms of what we have

so x equal to 3 y equal to 7 and z equal to and

so the final solution that we expect is three seven eight in fact this is the solution of the given system

so the natural question is given a system of equations right

so given a system $ax = b$

so be the given system of linear equations now upon applying the

so let row b the elementary row operations that are performed on a and b now let me denote by let a dash or a tilde denote ρ of a and b tilde denote ρ of b now that i have a new system what is known as $a \tilde{x} = b \tilde{b}$

so this is the newly obtained system the given system is $ax = b$ and i have the new system $a \tilde{x} = b \tilde{b}$ the only relation that i have is that a is ρ equivalent to $a \tilde{b}$ that is you can obtain $a \tilde{b}$ from a just by applying the elementary row operations and similarly b is ρ equivalent to $b \tilde{b}$

tilde right that is a is a tilde is obtained from a just by applying the row elementary operations similarly for b and b tilde right

so b delta is obtained from a b just by applying the row elementary operations the only thing over here is that the same set of operations that you apply on a are also applied on b now the claim is that $a \times$ equal to b well the systems $a \times$ equal to b and a tilde \times equal to b tilde have the same set of solutions well how to prove this fact one thing that one will have to observe here is the following before we go into the proof of this

so lets try to observe some simple properties that if ρ is an elementary row operation it is just a single operation then ρ of a equal to ρ of i times e this can be easily observed just by applying all the three elementary operations just apply it on the a one can easily verify it at least for a three by three matrix although not difficult for general n by n matrix but it should be easy for a three by three matrix just apply it on a three by three matrix a and similarly apply the same elementary row operation on the identity matrix multiply it with a one can notice that these two are one and the same and because of this what one can observe immediately is the following in fact the following are the consequences first one is that well before i proceed

so let me say that let row one row two up to ρ s be a finite set of elementary row operations right i have a finite set of row operations let a be an n by n matrix then you apply all these finite set of elementary row operations one by one on a because of the previous one one can easily identify that this is going to be same as the same set of row elementary operations first applied on the identity matrix and then multiplied with the just the matrix a right

so this is just the matrix multiplication and here in the first one you are applying these s row elementary operations one by one on the identity matrix and then you are multiplying with the it with the matrix a second thing again one can easily observe is that if you are going to apply the same set of row elementary operations one by one on the matrix a this is going to be same as you apply each and every row elementary operation on the identity matrix in the same order and finally multiplied with e well when you apply ρ 1 on a this is going to be same as ρ 1 times identity

so ρ 1 of identity times e now when you are going to apply row 2 applied on ρ 1 identity times a what you will obtain is ρ 1 of identity time ρ 2 of identity times ρ 1 at a which is same as ρ 1 at identity times a

so simply by induction one can prove the remaining things right with these two things now a small observation or a note which i will just state here saying that a similar version also holds for n by m matrices also also holds for n by m matrices also first thing that one will have to observe right once you have observed this fact now one thing that one can easily notice is that an easy thing that again one can notice is that if a and b are any two if rows any row is an elementary row operation and if a and b are any two matrices which can be multiplied then a simple thing that one can observe is the following what is that ρ of a b is equal to ρ a times b well how does this follow just use the previous one ρ a times b sorry row row of a b this is going to be same as ρ of identity times row of identity times a b but row of identity it is again a matrix and we know that matrix multiplication is associative

so let us use all these things ρ of a b which is equal to ρ of identity times a b but use the fact that the matrix multiplication is associative and ρ ϕ is just a matrix

so this is ρ of i times a and then you multiply this with the matrix b which is same as but ρ of i times a this is just ρ a times b and therefore if ρ

so sorry if $\rho_1, \rho_2, \dots, \rho_n$ is a finite set of elementary row operations then the same conclusion as what we said earlier ρ_1 composed with ρ_2 composed with ρ_3 up to ρ_n if it acts on b this is going to be same as ρ_1 composed with ρ_2 up to ρ_n just let it act on a this matrix just multiplied with s once we have this now one can easily show that these two will have the same solution how is it possible to show that

so $Ax = b$ is the given system now suppose x is the solution to the system in fact above system then $\rho(A)x = \rho(b)$ where ρ is any elementary row operation we have this and therefore once we have this but use the previous one this is equivalent to saying that $\rho(A)x = \rho(b)$

so in the previous notation just we used just few minutes back this is same as $\tilde{A}x = \tilde{b}$ that is the solution x what we had for the system $Ax = b$ is also the solution for the system $\tilde{A}x = \tilde{b}$ now use the fact that the elementary row operations are invertible and therefore what we have is that if x is the solution for these system $\tilde{A}x = \tilde{b}$ then x is also the solution for the system $Ax = b$

so what we just did is only for the just one elementary row operation and now we know that a similar thing holds even for a finite sequence of elementary row operations when applied altogether and therefore the system $Ax = b$ and the system $\tilde{A}x = \tilde{b}$ have the same solution right

so whatever we have said thus the systems $Ax = b$ and $\tilde{A}x = \tilde{b}$ have the same set of solutions if \tilde{A} and \tilde{b} are obtained from A and b respectively just by applying a finite set of elementary row operations right just by applying a finite set of row operations one can obtain a new system $\tilde{A}x = \tilde{b}$ we just concluded is that the system $Ax = b$ and the system $\tilde{A}x = \tilde{b}$ have the same set of solutions now let us do one more problem $2x - 3y = -21$ $3x + 2y = 18$ $x - 5y = -49$ one can notice that this is an over determined set of equations over determined system right this is an over determined why do we call it as an over determined system because what we have is just only two variables x and y but we have three equations now let us try to solve the system let us first write the augmented matrix or before that let us first write the matrix form $\begin{bmatrix} 2 & -3 & -21 \\ 3 & 2 & 18 \\ 1 & -5 & -49 \end{bmatrix}$ evaluated at x, y should give us minus twenty one one and minus fourteen this is the system that we have now let us try to write down the augmented matrix $\begin{bmatrix} 2 & -3 & -21 & | & 0 \\ 3 & 2 & 18 & | & 0 \\ 1 & -5 & -49 & | & 0 \end{bmatrix}$ lets augment it with the constant of matrix minus twenty one one minus forty nine this is the given augmented matrix or this is the system and we have given system we have write down in the as the augmented matrix first thing that we will have to do is that look for the leading coefficient

so this is the leading coefficient and we will have to convert it into one

so let us do that r_1 is replaced by r_1 one by two times r_1

so what you will have is one minus three by two we are augmenting it with minus twenty one by two

so let us have the other things as it is three two one eight minus five minus forty nine

so the next thing that will have to do is that convert the remaining elements of the first column into zero replace r_2 or r_3 by $r_2 - 3r_1$ and similarly replace r_3 by $r_3 - 8r_1$ $r_1 - 3r_2$ lets augment it with minus twenty one by two first one second one we have zero i have again zero here what we will have is 2 minus

so plus 9 by 2 1 plus 3 times of this which is sixty three by two and again minus five we are multiplying this with eight plus twenty four by two and here you will have minus forty nine plus twenty one into eight one sixty two by now let us try to write down the final one

so the matrix that we finally obtain here in the stage is one zero zero minus three by two thirteen by two fourteen by two augmented with minus twenty one by two sixty five by two eighty one one fifty sorry one thirty yes because one sixty two by two which is eighty one eighty one sorry eighty one minus forty nine ok ok eighty one minus forty nine which will give us eleven minus nine which is twelve seven minus four three

so we will have thirty two ok it will have thirty two now the next one that will have to look at is this term which we will have to convert it into zero if you start conv

so what we will have to for you this will have to convert into one r two should be replaced by two by thirteen times r two one zero zero minus three by two you will have one

so the remaining rows are untouched you have twenty one by two which is five thirty two i will have to convert the other things into zero

so what i will do is r one is replaced by three by two times r two plus r one and similarly r three is replaced by minus fourteen by two times r two plus r three what will i have one zero zero zero one zero and then i will have to work out here minus twenty one by two right five into three by two which is fifteen by two minus twenty one by two and similarly i will have five here and the last one thirty two minus seventy by two which is thirty five yes this should be thirty five exactly yes they should be thirty five yeah

so the previous one this should be one sixty eight

so that you get it as eighty four plus three

so this is going to be thirty five yes

so the final resulting matrix that we have is this is the final resulting matrix one zero zero one zero zero and then what you will have is minus three five zero

so now let us write down the final set of equations x equal to minus three y equal to five and z equal to sorry there is no z sorry right

so this is the solution to the system now if you notice this one the last row completely turns out to be zero now if at all there is a system if at all there is a system in which when you reduce the a matrix to its row echelon form row reduced echelon form and if the last row turns out to be zero but if you apply the same set of row elementary operations on the constant matrix and notice that the last term or the whatever be the wherever the zero sort here and if you obtain a non-zero term then one can easily conclude that such a system has no solution or let us just make a note in terms of rank a system of linear equations has a solution if rank of the given matrix or the coefficient matrix a is equal to the rank of the matrix a augmented with b the constant matrix b if these two the ranks of these two matrices if they coincide then you say that such a system has got a solution if they doesnt possess then we say that such a system has got no solution now lets do one more example two x minus three y plus two z equal to thirteen three x plus y minus z equal to two three x minus four y minus three z equal to one this is the given system now first let us try to write down the matrix form of this two minus three two three one minus one three minus four minus three when applied on the system when applied on the unknown xyz should give me thirteen two and one now first as usual let us write down the augmented matrix two three three minus three one minus four two minus one minus three augmented with thirteen two the coefficient matrix when we adjoin the constant of matrix along with the coefficient matrix we call it as the augmented

matrix we first look for the non zero rows and there are no nonzero rows in this case and therefore what we find is we look for the fourth non-zero coefficient the first row which is going to be just two we convert it into one r_1 one is replaced by r_1 one one by two times r_1 one one the remaining rows are untouched minus three by two one one minus one minus four minus three and then here it is minus third third sorry thirteen by two two one now we will have to convert the other elements of the first column into zero let us do that this will replace r_1 one by r_1 one plus three by two times r_2 and r_3 by r_3 minus half times r_1

so what we will have

so the first column will will have one zero zero second column again we will have zero one zero now let us compute the third one r_1 one r_1 one which is one plus three by two

so which is will have minus eight by minus ah

so we will have twelve by eleven second row remains unchanged minus eight by eleven third row r_3 minus six ah plus minus half times eight by eleven

so plus four by eleven augmented with the last column thirteen by two plus three by two times minus thirty five by eleven

so you will have

so minus one zero five upon eleven second row the second one remains unchanged third one minus thirty seven by two plus thirty five by twenty two

so the resulting matrix in this case is one zero zero zero one zero one minus twelve by eleven

so you have minus one by eleven minus eight by eleven second one minus twenty minus sixty six plus four by eleven plus four

so you will have minus sixty two by 11 the last column you will have 13 into 11 which is 143 minus 105

so you will have it as 38 by eleven ah sorry minus thirty eight by twenty two second one remains unchanged it is just this one the last one which you will have is ah thirty seven into eleven which is four naught seven

so four naught seven plus ah minus four naught seven plus thirty five which will give you three seventy two

so this is going to give you three seventy two upon eleven sorry upon twenty two

so the last element here that is the minus sixty two by eleven we will convert this into one

so r_3 is replaced by eleven minus eleven by sixty two into r_3

so what we will have here is one zero zero zero one zero minus one by eleven minus eight by eleven one the last column thirty eight by twenty two minus thirty five by eleven and here you will have ah this is going to give me ah minus three seventy two and minus 60 upon minus 62 that will give me just 6 and 11 by 22 will give me half

so i will have just 3.

so what we will have to do now is convert this minus one by eleven and minus eight by eleven into one sorry into zeros

so let us do that now r_1 one is replaced by r_1 one plus one by eleven times r_3 and similarly r_2 is replaced by r_2 plus eight by eleven times r_3 let us compute it the first and second columns they remain unchanged one zero zero zero one zero and based on these computations it is again clear that the last column is again zero zero one let us compute the final column which is the augmented one what we will have is r_1 one thirty eight by twenty two plus one by eleven times r_3 which is three by eleven the second one r_2

so minus thirty five by eleven plus eight by eleven times r_3

so which is twenty four by eleven and the last term is just three
so the resulting matrix here is one zero zero zero one zero zero zero one
so thirty eight plus thirty eight plus six which will give me forty four forty
four by twenty two will just give me two the other one minus thirty five plus
twenty four which is just minus eleven upon eleven
so i will have just one and the last one is just three therefore the solution
is two minus one three is the required solution with this let us stop this
lecture in the next lecture we will see some more examples on solving eq system
of linear equations especially on those systems which have no solution and which
possess infinite number of solutions thank you all you

Prutor@elitk