

welcome students welcome back in the last lecture on the series of lectures on matrices we saw what are known as elementary row operations and we saw how to use it to obtain what is known as the row reduced echelon matrix in fact in the last towards the end of the last lecture we saw an example now let us do one more example let the matrix A equal to $\begin{bmatrix} 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ zero zero zero four two zero this is the matrix that we have first thing that you will have to notice is that we have a zero row

so let us push it to the last one r_3 is swapped with r_4 what we will have is $\begin{bmatrix} 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ we have this look for the first non zero call row which is the first one and the first non zero element is this three

so let us try to convert into one r_1 is replaced by one by three times r_1 what we will have a zero one by three zero sorry zero one zero one by three other rows remains unchanged now let us convert the other element which is in that same column which is four into zero r_3 is replaced by r_3 minus four times r_1 first row remains unchanged zero one zero one by three second row again remains unchanged zero zero four one third row we are making a change zero minus four times zero which is zero four minus four times one which is zero again two minus four times zero which is two zero minus four times one by three you have minus four by three the last row remains unchanged and then the next one is let us look at this sub matrix this part

so the first one is non zero rows this and the non zero element is four

so let us convert it into one r_2 is replaced by one by four times r_2 zero one zero one by three zero zero one one by four zero zero two minus four by three zero zero zero zero now you have a z_2 here

so let us try to convert this two into zero r_3 is replaced by r_3 minus two times r_2 zero one zero one by three zero zero one one by four zero zero two minus two times one which is again zero r_3 minus four by three minus two times r_2 which is half zero zero zero zero and

so the resulting matrix here is zero one zero one by three zero zero one one by four zero zero zero

so minus four by three minus half which is minus eleven by six finally you have

so you have to consider this part you have a non zero force non zero element is this eleven by six will convert minus eleven by six will convert this into one r_3 is replaced by minus six by eleven times r_3 zero one zero one by three zero zero one one by four zero zero zero one the remaining things are going to be zeros and then now i have one by four and one by three here i will have to convert them into zeros r_1 is replaced by r_1 minus one by three times r_3 and similarly r_2 is replaced by r_2 minus one by four times r_3 three zero one zero all three things remains unchanged only thing is that

so one by three minus one by three times one which is zero similarly the next one zero zero one one by four zero zero zero one sorry this is zero this is not one by four you have zero zero zero zero zero

so this is the thus the row reduced echelon matrix of A obtained by applying the row elementary operations is the following matrix $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and the last row is $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ now let us do one more example let the matrix A $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ minus 2 3 minus 4 two five let us try to convert this into its rre

so the first row is non zero and the first element the non zero element is one so let us not bother

so what we will have to do is convert this minus four into zero

so will replace r_2 by r_2 plus four times r_1

so the first row remains unchanged 1 minus 2 3 second row minus 4 plus 4 times r_1

so minus 4 plus 4 times 1 which is 0 2 plus four times minus two

so what you will have here is minus six last one five plus four times three

which means five plus twelve

so i will end up with seventeen now let us look at the next row

so next to non zero one is this minus six seventeen which is a one by two matrix

so we will have to convert this minus six into one r_2 is replaced by minus one by six times r_2 the first row remains unchanged one minus two three zero one minus seventeen by six our next aim here is we have this minus two which we will have to convert this into zero

so let us do this

so what will i do i will replace r_1 by r_2 sorry r_1 plus two times r_2 i will have one zero in the next year i want 0 here and i will have my $1 \ r_1$ plus 2 times r_2 which means 3 minus 17 by 3 and i will have minus seventeen by six

so what is three minus seventeen by three which is nine minus seventeen by three

so nine minus seventeen you will have eight

so minus eight

so this will end up with one zero minus eight by three zero one minus seventeen by six this is the matrix that final h along rho reduced echelon matrix fine having said about how computing how to compute the row reduced epsilon matrix lets try to find out some of its applications or look at some of its applications to it the first application that we will have is in finding what is known as rank of a matrix how does one define the rank of a matrix

so the rank of a matrix is defined as the number of non zero rows in its row reduced h alone matrix you reduce the given matrix into a row reduced to hlr matrix look at the number of non zero rows and that is the rank of the matrix now the question is given a matrix a how many row reduced echelon matrices are possible if it is unique then this rank makes sense then the definition that we have given makes sense if it is not unique then what to do if you are going to have two then which one should i consider all these problems occurs in fact

so let me just make a note and i will not going to the details of this of course this note will be intuitively clear why it is true

so let me just remark or make a note saying that given a matrix a then the row reduced echelon matrix associated with a is unique which means given a matrix a there is only one row reduced epsilon matrix associated with this that is the procedure that we have given just few minutes back right that gives you the unique row reduce matrix or the row reduced echelon matrix associated with this matrix a and thus the definition of the rank of a matrix makes sense now let us proceed to calculating the rank of the matrix well let us look at the two both the examples that we saw the first example that we had just few minutes before

so let us just look at the examples the first one that we had is zero zero 4 1 0 3 0 1 0 0 0 0 0 4 2 0 this was the matrix that we had and the row reduced $r_r e$ right will just call it as $r_r e$ right this is the matrix that we have and the $r_r e$ corresponding to this is 0 1 0 0 0 0 1 0 0 0 0 1 and 0 0 0 0 this was the matrix that we have

so the number of non zero rows in the $r_r e$ row reduced epsilon matrix of a is three therefore rank of the matrix a is just three let us look at the next example that we had second example our matrix was this one minus two three minus four two and five this was the matrix that we had and the $r_r e$ or the row reduced epsilon matrix corresponding to this was one zero minus eight by three zero one and minus seventeen by six this was the $r_r e$ that we had

so the number of non zero rows in this $r_r e$ is two therefore rank of a is now let us do one more example let us choose a as two three four five two one this is the matrix that we have now let us try to convert this into its $r_r e$ if you

look at the look at this matrix there are no zero rows

so let us not bother about it first let us look at the non zero call first non zero column first non zero column is the first column itself and the first non zero core element in that first column is the first element which appeared in the first row itself and which is two now let us make this into one

so we are going to replace r_1 by half of r_1 let me write it here r_1 is replaced by half of r_1 you have one three by two four five two one next thing that you will have to do is will have to convert this four and two into zero

so i will have to replace r_2 by r_2 plus minus four times r_1 and r_3 three by r_3 plus minus two times r_1 i have one and three by two i will have to convert these two into zero right therefore these two operations are done

so now let us calculate the remaining five plus minus four into three by two

so minus four into three by two which is minus six

so five minus six it is minus one the next one one plus minus two into three by two

so minus two into three by two which is minus three

so one minus three which is minus two right

so let me delete the first row and the first column what i will end up with is the this sub matrix two by one sub matrix which is itself a non zero non zero there is no non zero row and there is no and you do not have the force non zero element appears in the first one

so let me first make that into one r_2 is replaced by r_2 sorry r_2 is replaced by minus one into r_2 one three by two zero one zero minus two i have this then i will have to make the other two elements into zero i will replace r_1 by r_1 plus minus three by two times into r_2 and similarly r_3 is replaced by r_3 plus two times r_2 right

so these two are going to remain unchanged and i will have zero one see this is the epsilon form that i have and the number of non zero elements non zero rows in this rre row reduced echelon matrix is two therefore rank of the given matrix a is let us do one more example may be the fourth example one two zero zero zero zero zero zero zero one three this is the matrix that i have

so you have a three by four matrix let us try to calculate the rank of this matrix the you have a zero row which is the second row which is above a non zero row

so let us push it to the last one is swapped with r_3

so what will i have one two zero zero zero one three zero zero zero the first non-zero column is this you have the first non-zero column and the first non-zero element is itself one

so i am not bothered about this and it is also one

so i am not bothered and the other elements in that column are also zero

so only thing that i will have to do is look at the other things let me delete the remaining one and let me look at the remaining sub matrix which is a two by three matrix and now the first non-zero column appears here which is one and zero and again if you look at this one one and zero this one and zero what what is happening here one and zero this one this one and zero this is the first non-zero column and the first entry is itself one and the other entry is in that column or other entries in that column or zero

so let me remove the first and second first column and first row and the remaining one is just the zero

so i finally just by swapping i have ended up with a row reduced echelon matrix

so this is the row reduced echelon matrix obtained from the given matrix the number of non zero rows in this matrix and this r r e is two therefore rank of the matrix yes having said about the rank the next next application that will we

are going to look at is what is known as invertibility of a matrix

so a square matrix A is said to be invertible if there exist again a square matrix B with order of B equal to order of A such that $AB = BA = I$ equal to the identity matrix right if you just multiply A with B or B with A it should give you the identity matrix of the same order then if you can obtain such a matrix B then you say that the matrix A is invertible now how to verify whether a matrix is invertible or not and this can be done by using what are known as the row elementary operations

so how to obtain this in fact

so let me just state a result suppose R is the row reduced echelon matrix obtained from the given matrix A given square matrix let me say this then A is invertible if and only if the row matrix R is the identity matrix how to check whether a matrix is invertible or not just find out its row echelon form once you know that the row array that you have finally obtained is just the identity then such a matrix has to be invertible the matrix B is called as the inverse of A the question is this is the inverse of a matrix unique is the inverse of a matrix unique in fact the answer is yes and how to prove it suppose that A is an invertible matrix let B and C be their inverses that is

so $AB = I$ and $CA = I$ and $AC = I$ and $BA = I$

so let me call the first one as one and the second one as two now let us observe a simple thing given any square matrix A $A \cdot I$ is going to be just A why is it

so let us prove it let $A_{ij} = a_{ij}$ and $I_{ij} = \delta_{ij}$ let me write it as $(A \cdot I)_{ij} = \sum_k a_{ik} \delta_{kj}$ and what are these δ_{kj} that is going to be one if $k = j$ zero if $k \neq j$ this is what you have

so now let's multiply A and I

so $(A \cdot I)_{ij} = \sum_k a_{ik} \delta_{kj} = a_{ij}$ which by the definition of matrix multiplication summation k running from one to n

so let us assume that the matrix A is an n by n matrix

so you have k running from 1 to n $a_{ik} \delta_{kj}$ now δ_{kj} these are the entries of the identity matrix and therefore this is one only if k and j are one and the same

so unless $k = j$ right this is going to be zero

so the only term that survives is a_{ij} well the all other terms are going to become zero

so what $A \cdot I$ will have is just A right and $I \cdot A$ is one

so I will just end up with A this is just the matrix A and similarly what will I have

so let me just write down therefore $A \cdot I = A$ let me just mark it as (1) similarly $I \cdot A = A$ we call it as (2) therefore let me start with $B \cdot A = I$ this is from (1) but $I = B \cdot A$ right this follows from (2) which is same as $B \cdot A \cdot A^{-1} = I \cdot A^{-1}$ by associativity of matrix multiplication first one that we used is the equation (1) second one that we used is equation (2) and finally what we have used is the associativity of matrix multiplication but from the equation (1) what will I have I will have $B \cdot A \cdot I = I \cdot I = I$ but $I \cdot A \cdot A^{-1} = I$ this follows from equation (2) thus $B = I \cdot A^{-1} = A^{-1}$ what we have shown is that the inverse of a matrix if at all it exists then it is going to be unique let us verify one more property for the inverse for any two square matrices A and B of same order the invertibility of A and B implies the invertibility of $A \cdot B$ and the inverse of $A \cdot B$ is equal to $B^{-1} \cdot A^{-1}$ right

so the unique inverse that we obtained

so once you know that a matrix A is invertible right then you know that the inverse is unique then we will denote that inverse by A^{-1} that is what we have the inverse of A is A^{-1} right and it is

given by $b^{-1} a^{-1}$ now let us go with the proof of this how to verify that something is the inverse

so the inverse of $a b$ is $b^{-1} a^{-1}$ well let us verify that $a b$ times $b^{-1} a^{-1}$ is the identity and similarly $b^{-1} a^{-1}$ times $a b$ is identity and by the uniqueness this should be the identity for $e b$ sorry this should be the inverse for $a b$ $a b$ times $b^{-1} a^{-1}$ which is equal to a times $b b^{-1}$ times a^{-1} but $b b^{-1}$ is just the identity

so this is same as a times identity times a^{-1} but a times identity is just a times a^{-1} which is going to be identity similarly $b^{-1} a^{-1}$ times $a b$ which is equal to $b^{-1} a^{-1} a$ times b and $a^{-1} a$ is the identity

so i will have $b^{-1} i$ times b but i times b is just b

so this is same as $b^{-1} b$ which is going to be identity

so the inverse of $a b$ is which is $a b$ whole inverse which is given by $b^{-1} a^{-1}$ now having said all these let us now compute the inverse of a matrix using the row elementary operation before that

so let us just see how to compute it what we observed is the following what is that that we observed suppose that you have a matrix a which is invertible then the row reduced echelon matrix associated with that matrix a is the identity matrix

so few observations what is that

so if ρ is an elementary row operation on the set of all matrices then ρ is invertible as a function that is ρ is both one one and one two right which means that you get another function g such that ρ times g is equal to g times ρ which is equal to identity on the space of matrices in fact the last set of lectures on sets and functions we observed that if you have a function which is one one and onto then such a function can be inverted

so similar

so the same thing is happening here that is the every row elementary operation can actually be inverted

so if you observe the first row elementary operation which is the a_h interchanging of two rows that is an invertible operation because its inverse is again interchanging the same set of rows the second one multiplying a given row say eighth row by a scalar α the scalar λ or α then the inverse operation is multiplying the same i throw by the scalar $1/\alpha$ of that scalar say λ or α right similarly the third operation the i th row is replaced by the i th row plus a scalar times scalar λ times the r th j th row right now that is again invertible

so if ρ_1 row two etcetera ρ_m are the elementary operations performed on a matrix on an invertible matrix or the elementary row operations elementary operations performed on an invertible matrix a to obtain the identity matrix rather than starting with if suppose or the elementary operations performed on an invertible matrix a to obtain the identity matrix then what do i have ρ_1 composed with well i will write it as $\rho_m \rho_{m-1} \dots \rho_1$ when if it is performed on a what you get is the identity matrix

so thus ρ_m composed with ρ_{m-1} up to ρ_1 composed with ρ_1 when you apply this on the identity and multiplied with the matrix a what you obtain is the identity matrix

so this will imply that the inverse of the matrix a is ρ_m composed with ρ_{m-1} applied on identity is the inverse

so how to obtain the inverse of a matrix just apply the same set of elementary operations that you applied on a to obtain the identity matrix apply the same set of operations on identity what you will end up with is the inverse of a now let us do an example a equal to $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ this is the matrix that we

have

so let us try to find the inverse of this

so what we will have to do is write down the matrix A and along with this let's write down the identity matrix now let us try to convert this part the portion that corresponds to A into the identity matrix simultaneously applying the same operations on the identity matrix

so once you know that this part is has converted into identity matrix we know that it is invertible and then the one that you have obtained on the other end is the inverse of it and the final row echelon matrix that we have obtained if it is not the identity matrix then we can conclude that the matrix is not invertible well let us try to convert this into the row reduced echelon matrix into its row echelon form

so you have the first non-zero row is the first one and the first non-zero element is one

so let us not bother let us look for the other ones the other ones we have one and one

so let us convert them into zeros $r_2 - r_1$ and $r_3 - r_1$ first row remains unchanged second row $r_2 - r_1$ one minus one which is zero two minus one which is one one minus one which is zero zero minus one minus one one minus one one zero minus zero zero again $r_3 - r_1$ one minus one which is zero two minus one which is one three minus one which is two zero minus one minus one zero minus zero zero one minus zero which is one now let us look at the next non zero row which is the second one and now let us try to look at this one you have zero one zero

so what we will rather do is

so we will have to convert this into one which is already one

so let's not bother let's convert try to convert this one and one into zeros $r_1 - r_2$ is replaced by $r_1 - r_2$ r_3 is replaced by $r_3 - r_2$ $r_1 - r_2$ one minus zero which is one one minus one which is zero one minus zero which is one the other part one minus minus one which is two zero minus one which is minus one zero minus zero which is zero second row remains unchanged zero one zero minus one one zero third one $r_3 - r_2$ zero minus zero which is zero one minus one which is zero two minus zero which is two minus one minus minus one which is zero zero minus one which is minus one one minus zero just one finally you have two let us convert this two into one r_3 is replaced by half of r_3 one zero one two minus one zero zero one zero minus one one zero zero one zero minus one one sorry zero minus half and half

so what you will have is now we will have to convert this one into zero $r_1 - r_3$ is replaced by $r_1 - r_3$ one zero zero $r_1 - r_3$ two minus zero which is two minus one minus minus half

so which is minus one plus half

so you have minus half zero minus a half you again have minus half the other things remains unchanged zero minus half and half thus what we have obtained is the identity matrix which is the row echelon matrix row reduced echelon matrix of the given matrix A and therefore the given matrix is invertible and once you know that it is invertible the matrix that you obtained on the right hand side is the inverse of this

so thus $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{bmatrix}$ with this will stop this lecture in the next lecture we will see some more applications row elementary operations especially on solving systems of equations thank you you