welcome students welcome back to the series of lectures on matrices in the previous lecture we introduced the concept of multiplication of matrices and we saw some of the properties
 so let us proceed further by doing some more properties on the multiplication of matrices first one matrix multiplication is associative that is for any three matrices a b and c such that a and b are compatible for multiplying and b and c are compatible for multiplying then a dot b dot c is equal to a dot b dot c
 so let us prove this
 so let a be ah n by m matrix b b m by r matrix and c b ah r by s matrix
 so this is possible because
 so this is possible because of the given hypothesis
 so this kind of choice for n and m n m r and s right
 so this is the hypothesis that we have a and b are compatible for multiplying and similarly b and c are compatible for multiply
 so what you have is that a is an n by a matrix b is an m by r matrix and c is an r by s matrix
 so let a equal to a i j right 1 less than or equal to i less than or equal to n 1 less than or equal to j less than or equal to m b equal to b i j one less than or equal to i less than or equal to m one less than or equal to j less than or equal to r and finally c equal to c i j one less than or equal to i less than or equal to r and one less than or equal to j less than or equal to s now let us try to calculate a dot b dot a dot b dot c is a dot b is this matrix b i j a dot with c i j fine now if you multiply these two the resulting matrix will have the following thing k running from one to r b i k c k j
 so one should notice carefully over here right that these two are compatible therefore this makes sense which is equal to a i j with this matrix whose entries are given by summation k running from one to r b i k c k j right now let us try to expand it out let us again multiply these two equal to the entry is going to be summation t running from one to m a i t
 so the remaining is going to have what you want is the ah maybe if i am going to call the product of b and c as the matrix d with entries d i j
 so what i needed is d t j right this is the i j th element i wanted
 so where d t j is the t j th entry in the matrix b dot c
 so this is going to be equal to summation t running from one to m a i t and d t j let us write it down summation k running from one to r b t k c k j this is what i have which is equal to summation t running from 1 to m summation k running from 1 to r a i t b t k c k j this is what i have have finally after expanding all the brackets i have this
 so the finally a dot b dot c the i j th entry is given by this formula now let us try to calculate the other one a dot b dot with c which is going to be the product of matrices with entries a i j times the product of times the matrix with entries b i j times with c
 so this we know by product rule the entries are given by summation k running from one to m a i k b k j a i k b k j
 so dot with c equal to let me write down summation k running from 1 to m a i k b k j these are the entries dot with c i j which is going to be equal to the i j th element
 so i will have to multiply let me just use the same thing summation over t running from one to r i need the i th element element
 so summation t running from one to r
 so the i th element is summation k running from one to m a i k b k t this is the i th element this whole thing multiplied with i will have to multiply it with the t jth element of this which is exactly the c t j this is what i have which is equal to summations t equal to one to r summation k equal to one to m a

i k b k t c t g right now what you will have to notice is the following will have to notice that this is same as what we had the expression for a dot b dot c
 so how to compare it
 so what we will do in this case is let us interchange the roles of t and k or because they are just the dummy indices
 so i will just write it as replace t by k and k by t k equal to one to r and t equal to one to m i have replaced k by t a i t and t by k b k t sorry b t k yes and c k j this is what i have now just look at the expression that we had last time and you can notice that this is exactly same as a dot b dot c thus we have shown that the matrix multiplication is associative next one natural question that one will ask is about commutativity the truth is that matrix multiplication is non commutative in general
 so in fact let us do an example let us look at an example for this let a equal to 1 minus 2 3 minus 4 2 5 and b equal to 2 4 2 three five one first let us try to compute a b a b equal to 1 minus 2 3 minus 4 2 5 times this b 2 3 4 5 2 1.
 equal to one into two two minus two into four minus eight three into two six
 so plus x one into three minus two into five
 so minus ten three into one plus three minus four into two minus eight two into four plus eight five into two ten plus ten minus four into three minus twelve two into five plus seven five into sorry plus ten plus five into one which is five the resulting matrix is 2 minus 8 minus 6 plus 6 which is 0 3 minus 10 minus 7 plus 3 which is minus four eight minus eight zero plus ten which is ten minus twelve plus ten which is minus two plus five which is three on the other hand let us try to compute ba ba equal to 2 3 4 5 2 1 times 1 minus 2 3 minus 4 2 5 equal to 2 into 1 to 3 into minus 4 minus 12 2 into minus 2 which is minus 4 3 into 2 6
 so plus x 2 into 3 which is 6 five into three fifteen
 so plus fifteen four into one five into minus four which is minus twenty four into minus two which is minus eight five into two ten four into three which is twelve five into five which is twenty five
 so plus twenty five two into one one into minus also minus four two into minus two which is minus four one into two which is two two into three which is six plus five one into five which is five the resulting matrix is two minus twelve which is minus ten minus four plus six which is minus which is two six plus fifteen which is twenty one four minus twenty which is minus sixteen minus 8 plus 10 which is 2 12 plus 25 which is 37 2 minus 4 which is minus 2 minus 4 plus 2 again minus 2 6 plus 5 which is 11.
 you can notice the following that a b is a matrix of order two by two while ba is a matrix of order three by three and also the entries of a and b cannot be matched
 so once you have that these two are of matrices of different order these two cannot be compared therefore a b is not equal to b a natural question is how about ah square matrices of same order let us do look at another example let a equal to 1 2 3 and 4 and let us choose b as five six seven and eight now let us try to compute a b a b which is 1 2 3 4 times 5 6 7 and 8 which is equal to one into five five seven into two fourteen
 so plus fourteen one into six six two into eight which is sixteen three into five fifteen seven into four twenty eight 3 into 6 which is 18 plus 4 into 8 which is 32 the resulting matrix is 5 plus 14 which is 19 6 plus 16 which is 22 15 plus 28 which is 43 18 plus 32 which is 50.
 now let us try to calculate b a b a equal to 5 6 7 8 times 1 2 3 and 4 which is equal to 5 into 1 5 plus 6 into 3 which is 18.
 five into two ten six into four twenty four seven into one seven eight into three twenty four seven into two fourteen eight into four thirty two which is

equal to five plus eighteen which is twenty three ten plus twenty four which is thirty four 7 plus 24 which is 31 14 plus 32 which is 46 thus a b which is equal to 19 32 43 and 50 and you can notice that not even a single entry is equal to the entries of the matrix b a 23 34 31 and 46 which is same as right therefore a b is not equal to b a even when a and b are square matrices of same order the next one next property that matrices in general processes is the following

so for example if alpha and beta or any two scalars such that alpha dot beta when you multiply these two and if the resulting zero scalar then either alpha equal to zero or beta equal to zero one of these happens but this does not happen in the case of matrices

so let us look at a simple example where this fails

so let a equal to zero minus one zero two and b which is three five zero zero you have this you have these two now let us try to compute a b zero minus one zero two with the matrix three five zero zero going to be zero into three which is zero plus minus one into zero you will have zero zero into five you have zero minus one into zero

so it is again zero zero into three it is zero plus two into zero which is again zero zero into five it is zero plus two into zero will half zero

so the resulting matrix is just the zero matrix

so matrices although they look like as if they behave like real numbers or complex numbers they have their own set of properties that is they are highly non-commutative and similarly even if you have two non-zero matrices their product can lead to this 0 matrix now let us move on to the concept of what is known as or elementary row operations and row echelon matrices let us first begin with defining what is a row echelon matrix and then try to understand what is known as zero operations fine

so let us begin with the definition a matrix is called row reduced echelon right or a row echelon matrix if the following properties hold first one every zero row is below every non zero rho second one the leading coefficient what does it mean by a leading coefficient first non zero coefficient of every row is one well the third one a column which contains a leading non-zero entry contains the leading because the second one says that the leading coefficient should be one

so which contains the leading coefficient and one of a row right if you have a column which contains the leading coefficient one of a row has all the other coefficients equal to zero right

so you have three now the last condition fourth one suppose the matrix has r non zero rows that means that the remaining rows are zero right matrix has are nonzero rows if the leading nonzero entry leading non-zero entry of the ith row occurs in the k i th column then what can you say about k 1 k 2 k r then k 1 is strictly less than k 2 up to strictly less than k these are the four things that you need to define the notion of what is known as the row reduced echelon matrix now let us do some examples first one let us look at this one one zero two zero zero zero zero one zero the first condition says that every zero row is below every non zero row

so second row is a zero row but it is above a non zero row right the second row is above a non zero row the third row is non zero and because what we want is that every zero row should be below every non zero row and the second row which is a zero row is above a non zero row and hence not row reduced and not a row reduced echelon matrix right let us look at another example second one the same thing

so lets swap the second and third rows one zero two zero zero zero zero one zero

so you have zero here one zero one one two zero right right

so this is the matrix that you have one zero one zero two zero zero zero zero
 so first one the zero row here is the last one which is below all the nonzero other nonzero rows second one the first leading coefficient should be in every row should be one which is the first non-zero in the first row this is the first one first leading coefficient which is non-zero this year the first row you have one which is fine but in the second one you have two
 so the first leading coefficient or the first non zero coefficient in the second row is two and hence not a row reduced echelon matrix let us look at another example let us look at this one 1 one two zero one one zero zero c the zero row which is the third one and it is below the other two zero rows or the other two non zero rows the first leading coefficient in the first row is the first element which is one and similarly for the second row it is the second element which is again one but if you notice the third condition that you need for a row reduced echelon matrix is that if wherever you have the first leading coefficient then the other other entries of that column should be zero
 so if you have one over here
 so the other entries are zero here but you have one which is the leading coefficient but then you have zero here but this is non zero right hence this is not a row reduced etchalon matrix let us look at one more 0 1 2 zero three zero zero zero five you have this
 so the zero row which is the third row which is below all the other rows second row
 so the first leading coefficient is here in the first row and the first leading coefficient is here in the second row all the all these are one and wherever you have the leading coefficient the other entries in that column are zero you have one in the first row second column and the other entries in that column are zero and you have one in the first the first entry in the second row is one right and the other entries are zero
 so k one
 so you have two non-zero rows
 so what you wanted is the kth column the fourth condition k one is one in this sorry two in this case which is the second column and k two in this case is one therefore what you have is k two is less than k one therefore this is this is not row reduced
 so this matrix is not row reduced let us look at another example 1 0 2 0 1 3 0 zero zero
 so you can notice that this zero row the third row is below all the non zero rows first thing second the leading coefficient or the first non zero coefficient in both first and second row leading coefficients in both first and second rows are just one right it is just one and therefore you are done the third one that you wanted is that all the elements right
 so the elements in a column containing a leading coefficient all the other elements right in a column containing a leading coefficient or zero
 so leading coefficient appears in first column and second column you can notice that apart from this one the remaining two elements are zero last one k one right the first one appears in the k one column or first column k one is one k two here it is secant two
 so k two is two
 so therefore k one is less than k two
 so thus this matrix is a row reduced h along matrix now the natural question that one would like to ask at this stage is is the following question given a matrix a is there any procedure to convert it into a row reduced echelon matrix this is the question
 so let me repeat the question given a matrix a is there any procedure to

convert it into a row reduced echelon matrix yes there exists a procedure and the procedure is to apply what are known as elementary operations or what are known as row elementary operations

 so let us first discuss what these row elementary operations are there are three row elementary operations first one multiplying the ith row by a non zero scalar say lambda

 so will denote this by the right ith row is replaced by lambda times r i lets do a simple example if you look at this matrix one two three four five six you just multiply the first row by two times of one what will you end up with you are multiplying the first row by two

 so two into one it is two two into two it is four and two into three is six four five six this is what you have

 so the new matrix two four six four five six is obtained from the first one by just multiplying the first row by the scalar two second one interchanging ith row and j through right this will denote it as the ith row and jth row are swapped or interchanged this is the notation that will have

 so let us look at one example 0 1 2 one zero three zero zero zero

 so this is an example

 so what is that that we are going to do we are just going to swap r one and r two and what will you have if you swap one zero three zero one two zero zero zero the third row elementary operation replace the i throw by the sum of ith row and a mu multiple of j throw you are replacing the ith row by r i the ith row plus a scalar times the j th row

 so we are replacing the ith row by r i the ith row plus the scalar mu times the j two this is what we are doing

 so let us do an example for this the matrix that you have is one two zero zero zero zero one three this is the matrix that we have now let us do the following r one is replaced by r one plus two times r two this is what we have we are just multiplying two times r two

 so first one is r one is one one plus two times zero which is one two plus two times zero will end up the with 2 0 plus 2 times 1 will end up with 2 0 plus 2 times 3

 so i will end up with 6.

 so 0 0 1 3 this is the one that we have right

 so these are the 3 elementary operations that we are going to perform to obtain what is known as the row echelon matrix now what is the procedure that we will have to apply to get a row reduced echelon matrix from a given matrix

 so procedure to obtain a row reduced echelon matrix from a given matrix

 so the following are the steps

 so let me just denote them by dots first step let me write it as step one because you will we may have to repeat certain steps again and again first thing that you will have to notice is that right every zero row is below every non zero that will have to verify

 so if that is not the case apply interchange of rows to push down the zero rows to the end of the matrix

 so once you do this what will happen by the end of this all the zero rows will be below every non-zero row now the second one step two find the first non zero column you will have to find the first non zero column in fact i should write it as from left right you will have to start from the left right

 so let us suppose that the first non zero column that it is k one right k one is the first non zero column step three again apply interchange of rows to push up

 so earlier we pushed down now we are pushing up to push up a row whose leading

non zero coefficient occurs in first non zero column column to the first row

so i want this to be in the first row if this coefficient is not one then what will i do divide the first row by the leading non zero coefficient divide the first row by the leading nonzero coefficient

so that the leading non zero coefficient becomes i want that to be one now step four next apply the last elementary operation r i replaced by r i plus mu times r j for suitable values of i sorry j and in fact i should write it as mu and r one right this should be r one for suitable values of i and mu

so that the first the first non zero column has non zero coefficients only in the first now step five repeat steps two two four for the sub matrix obtained by deleting the first row and first column until all the non zero rows are exhausted right until you finish all the non zero rows keep applying this again and again and again

so now let us do a simple example one one 1 2 1 1 2 3 this is the matrix that i have

so what are those things that we will have to do first step look if there are any zero rows and then apply interchange of rows to push them to the last one now if you look at this matrix there are no zero rows and hence you do not have to apply this first one second identify the first non-zero column which is the first column itself and then make sure that look at the first non-zero coefficient in that column where it appears the row in which chip it appears again it is in the first row first column

so there is no problem

so the next thing that will have to do is make the

so it is again 1

so we do not have to divide we will have to do is look at the other entries they are nonzero

so how to make it 0 replace r two by r two plus minus one times r one right r two is replaced by r two plus minus one times r one

so first row is kept as it is second row r two plus minus one times r one

so you will have zero here r two which is two plus minus one which is one one plus minus two which is minus one and then the last one again you replace r three by r three plus minus one times r one you will have zero

so again you will have one

so you will have three minus two which is one now what you do you look at this sub matrix one minus one one one look at this sub matrix and then do the same operations you do not have any nonzero things non zero rows and well you will have to look at the first non zero column which is this and the first nonzero coefficient is this one which is again one

so let us not bother

so we will have to convert the other things

so what i will do replace r one by r one minus minus one times r two

so i will have zero one minus one as it is i am just converting r one r one one minus minus one into zero which is like i will have one one minus minus one my one minus one which is zero two minus minus one into minus one which is one oh i will have to write it as plus psi it should be plus right

so two plus one which is three yeah again here last one r three is replaced by r three plus minus one times r two right

so r three

so i want this to be zero

so i will have zero here

so one minus one which is zero one plus minus one into one which is two

so i will have finally this is the one that i am the sub matrix which is a one by one matrix right

so this is only non zero coefficient and what i have is i should make this into one

so replace the last one is r three is replaced by one by two times r three i have other things one zero three zero one minus one zero zero one i will have to make the other two elements zero which is minus one and three

so let me just convert them into zeros

so what i will do is i will replace r one by r one plus minus three into r three

so what will i end up with i will have one zero zero and similarly i will replace r two by r two plus r three just r two plus r three

so what will i end up with i will just end up with zero one zero zero this is the thing that i have right thus the row reduced echelon matrix obtained after applying the procedure or the algorithm to the matrix one one two 1 2 1 1 2 3 is just the identity matrix and with this let me stop this lecture thank you all you