

welcome to the lecture on matrices and determinants in the last lecture we introduced the notion of a matrix and we saw some properties of the matrices especially we saw how to add matrices now let us look at the second one what is known as scalar multiplication

so we are going to multiply a real or complex matrix by a real scalar or a complex scalar let  $A$  be an  $n$  by  $m$  matrix that is  $A = (a_{ij})$  where  $i$  runs from 1 to  $n$  and  $j$  runs from 1 to  $m$  and let  $\alpha$  belong to  $\mathbb{R}$  set of real numbers or  $\mathbb{C}$  right

so by an  $n$  by  $m$  matrix the entries are just a real number or a complex number so we define the matrix  $\alpha A$  as follows

so  $\alpha A$  its  $ij$ th entry is given by  $\alpha a_{ij}$  right

so thus the matrix  $\alpha A$  is given by  $\alpha A$

so once you have this definition now let us look at some of the properties of this scalar multiplication first one that you can notice is that if you have  $\alpha$  and  $\beta$  or any two scalar then  $\alpha + \beta$  if you multiply it on  $A$  this is same as  $\alpha A$  and  $\beta A$  and then add these two matrices now how to do this how to give a proof of this

so now let us write down  $(\alpha + \beta)A$  well

so solution or a proof

so let

so because you have the matrix let me write the matrix  $A$  as  $(a_{ij})$  then by definition  $(\alpha + \beta)A$  this is going to be  $(\alpha + \beta)a_{ij}$  but we know that addition and multiplication they are distributive for real numbers

so this is same as  $\alpha a_{ij} + \beta a_{ij}$  and this is same as  $\alpha a_{ij} + \beta a_{ij}$  by the definition of addition of matrices  $\alpha a_{ij} + \beta a_{ij}$  this matrix plus the matrix  $\beta a_{ij}$  now again by the definition of the scalar multiplication this is same as  $\alpha a_{ij} + \beta a_{ij}$  which is equal to  $\alpha a_{ij} + \beta a_{ij}$  second one for any scalar  $\alpha$  and for any two matrices  $A$  and  $B$  of same order  $\alpha(A + B)$  is same as  $\alpha A + \alpha B$

so you are given that  $A$  and  $B$  are of same order

so let let me write  $A$  as  $(a_{ij})$  and  $B$  as  $(b_{ij})$  where  $A$  and  $B$  the matrices  $A$  and  $B$  are of same order

so  $\alpha(A + B)$  which is same as  $\alpha A + \alpha B$  let me write down in terms of entries

so that things will be

so let me use a small letter

so that things will be much clearer  $(\alpha A + \alpha B)_{ij}$  by definition of the matrix addition right the entry wise you add them

so it is going to give you the matrix  $(a_{ij} + b_{ij})$  now by the definition of the scalar multiplication this is going to just give you  $\alpha a_{ij} + \alpha b_{ij}$  again use the fact that addition and scalar multiplication they are distributive therefore this is same as  $\alpha a_{ij} + \alpha b_{ij}$

so one thing that you will have to notice in the second thing is that second equality that this holds because of the fact that  $A$  and  $B$  are of same order therefore you can add them right

so having said all these things we have finally come to the place where we have  $(\alpha A + \alpha B)_{ij}$  as the  $ij$ th entry by the definition of the addition of matrices this is same as  $\alpha a_{ij} + \alpha b_{ij}$  again lets use the definition of scalar multiplication this is same as  $\alpha(a_{ij} + b_{ij})$  the matrix  $(a_{ij} + b_{ij})$  plus  $\alpha$  the matrix  $(a_{ij} + b_{ij})$

so this is

so this matrix is with entries  $(\alpha a_{ij} + \alpha b_{ij})$  this is exactly  $\alpha(A + B)$

$\alpha \cdot \beta$  for any two scalars  $\alpha$  and  $\beta$ ,  $\alpha \cdot \beta \cdot a$  is same as  $\alpha \cdot (\beta \cdot a)$  which is same as  $\beta \cdot (\alpha \cdot a)$  well its enough to prove one of these things

so first one second because the third one follows because  $\alpha \cdot \beta$  is same as  $\beta \cdot \alpha$

so  $\alpha \cdot \beta \cdot a$

so as usual we assume that  $a$  is of the form  $a_{ij}$  which is  $\alpha \cdot \beta \cdot a_{ij}$   $\alpha \cdot \beta$  is again a scalar

so you are multiplying with the matrix  $a$

so by definition this is same as the matrix whose entries are multiplied by the scalar  $\alpha \cdot \beta$   $\alpha \cdot \beta \cdot a_{ij}$  notice the fact that the multiplication of scalars is associative

so this is same as  $\alpha \cdot \beta \cdot a_{ij}$  which is same as  $\alpha$  times  $\beta \cdot a_{ij}$  right you have  $\alpha$  times  $\beta \cdot a_{ij}$  which is equal to  $\alpha$  times the matrix  $\beta \cdot a_{ij}$  but then the matrix within the bracket is exactly  $\beta \cdot a_{ij}$  is equal to  $\alpha \cdot \beta \cdot a_{ij}$

so these are some of the properties of the scalar multiplication there is one more operation on the set of matrices is what is known as transpose of a matrix if  $a$  is any matrix then the transpose of  $a$  denoted  $a^t$  means transpose is obtained as follows right how do you obtain

so let me write  $a$  as  $a_{ij}$  and let  $b$  equal to or let me write  $a$  transpose this matrix as the entries as  $b_{ij}$

so then  $b_{ij}$  what are these this is given by  $a_{ji}$  these are given by  $a_{ji}$  this is the definition of the transpose

so once you have the transpose

so if  $a$  is a matrix of order  $n$  by  $m$  then the transpose  $a^t$  is a matrix of order  $m$  by  $n$  you will have the other way  $m$  by  $n$  well let us try to do an example let us try to calculate the transpose for some of the matrices first one let  $a$  equal to one by two one by root two one by root three three one by root three one by root five five one by root five one by root seven now let's try to calculate  $a^t$   $ij$ th entry of this is the  $j$ th entry of the corresponding matrix  $a$  or the underlying matrix  $a$

so the first one one month position

so we will have to look at again the one month position it is half second one what you have is one tooth position

so you will have to look at the corresponding entry in the two one position which is three again you have one third one three position

so you will have to pick the element from the three month position which is five again you have

so this is the two one position

so you will have to look at the one tooth position which is one by root two two tooth position one by root three two three three position

so look at the three tooth position which is one by root five again this is three one position

so look at 1 3 position in the corresponding matrix  $a$  you have 1 by root 3 1 by root 5 and then finally 1 by root 7 right this is the transpose of the matrix  $a$  lets do one more example let us write down a matrix with the complex entries with some complex entries  $i$

so where  $i$  denote the complex number  $2 + i$   $1 + 2i$   $3 + i$   $2 + 1 + 2i$   $3 + 4i$  plus five  $i$  three four this is the matrix now let us try to calculate a transpose first one it is exactly same as  $i$  second one the one tooth position

so  $i$  will have to look at the corresponding entry in the two one position which is three  $i$  now let me just write down this is four plus five  $i$

so the first column will be converted here as first row

so what you need is a second row

so you will have to correspond look at the corresponding second column

so two i two i three now i want the last row

so you will have to look at the corresponding last column one plus two y two plus three i four

so this is the matrix that we have now let us look at some simple properties  
soft transpose first one if you have two matrices of same order then  $a + b$  whole transpose is equal to  $a$  transpose plus  $b$  transpose proof  $a + b$  whole transpose

so let me write  $a$  as  $a_{ij}$  and  $b$  as  $b_{ij}$

so now  $a + b$  this makes sense only if  $a$  and  $b$  have got the same order and

so what will have to as mentioned earlier what you will have to assume is that  $a$  and  $b$  should have the same order

so what we wanted is  $a + b$  whole transpose which is same as  $a_{ij}$  plus  $b_{ij}$  whole transpose which is equal to this matrix  $a_{ij}$  plus  $b_{ij}$  the whole transpose

so once you when you take the transpose  $ij$ th entry goes into  $ji$ th entry and the element in the  $ji$ th position goes to  $ij$ th position

so what you will have when you take the transpose you will end up with  $j_i$  plus  $b_j_i$  which is same as by the definition of matrix addition this is same as the matrix consisting of  $a_j_i$ 's plus the matrix consisting of  $b_j_i$ 's but this is same as  $a$  transpose and the next one corresponds to  $b$  transpose therefore  $a + b$  whole transpose is equal to  $a$  transpose plus  $b$  transpose second one for any scalar  $\alpha$  and any matrix  $a$   $\alpha a$  whole transpose is same as  $\alpha$  times  $a$  transpose proof

so as usual let me write  $a$  as  $a_{ij}$  then i wanted  $\alpha$  times  $a$  whole transpose which by definition is  $\alpha$  times the matrix with entries  $a_{ij}$  whole transpose which is same as  $\alpha$  times  $a_{ij}$  the whole transpose

so the  $ij$ th entry is given of this is given by  $\alpha$  times  $a_{ij}$  i need the transpose of this

so this is going to be the matrix with entries  $\alpha$  times  $a_{ji}$  which is equal to this when  $i$  same as my definition of matrix my scalar multiplication with a matrix  $\alpha$  times  $a_{ji}$  which is same as  $\alpha$  times  $a$  transpose right thus we have what we wanted fine let me go to the next definition a matrix  $a$  is called a symmetric matrix if  $a$  equal to  $a$  transpose similarly a matrix  $a$  transpose sorry matrix  $a$  is called a skew symmetric matrix if  $a$  equal to minus of  $a$  transpose by minus of  $a$  matrix i mean minus one times of that matrix right let us look at an example let us look at this one  $a$  is one two three 2 3 4 3 4 5

so let us first try to calculate  $a$  transpose

so one two three two three four three four five this is the one that we have obtained after taking the transpose

so note that  $a$  equal to  $a$  transpose therefore  $a$  is symmetric

so the next example let  $a$  be an upper triangular matrix

so what you have is an upper triangular matrix then  $a$  cannot be symmetric unless  $a$  is a diagonal matrix right for  $a$  to be symmetric you want the  $ij$ th entry and the  $ji$ th entry to be one and the same and for an upper triangular matrix you know that all the entries below the diagonal should be zero and therefore which means for an upper triangular matrix to be a symmetric matrix you want all the entries even above the diagonal should be 0 and that means that it should be a diagonal matrix

so in particular every diagonal matrix is symmetric now let us do some more properties well if  $a$  is any matrix then any  $i$  should i need a square matrix any square matrix then  $a + a$  transpose is also is a symmetric matrix right if  $a$  is any square matrix then  $a + a$  transpose is a symmetric matrix proof how to

prove this

so let  $A$  equal to  $A_{ij}$  its a square matrix then  $A$  transpose the entries are given by  $A_{ji}$  this is the the  $ij$ th entry of  $A$  transpose is  $A_{ji}$  now let us try to calculate  $A + A^T$   $A + A^T$  is given by the matrix  $A_{ij} + A_{ji}$  right

so  $A_{ji}$  means a matrix with entries  $ij$ th entries as  $A_{ji}$  but this again by the definition of the addition of matrices this is  $A_{ij} + A_{ji}$  this is what we have now let us try to calculate the transpose of this  $A + A^T$  transpose which is the transpose of the matrix whose  $ij$ th entry is  $A_{ij} + A_{ji}$  if you have a matrix  $A$  with entries  $A_{ij}$  then its entries the entries of a transpose is given by  $A_{ji}$  now i have a matrix whose  $ij$ th entries are  $A_{ij} + A_{ji}$  and therefore i will have to just swap  $i$  and  $j$

so the entries are given by  $A_{ji} + A_{ij}$  again note the fact that real number the addition of real numbers is commutative and therefore this is same as  $A_{ij} + A_{ji}$  which is same as  $A + A^T$

so let me call this as one from one right thus what do we have  $A + A^T$  this matrix is symmetric fine similarly if  $A$  is any square matrix then  $A - A^T$  is a skew symmetric matrix proof as usual let  $A$  equal to the matrix  $A_{ij}$  then the matrix  $A^T$  is given by  $A_{ji}$

so once you have this

so as usual lets start writing the matrix  $A + A^T$

so you have  $A + A^T$  if you look at the previous ones then this is going to be same as

so what we wanted is  $A - A^T$

so this is going to be  $A_{ij} - A_{ji}$  which is same as  $A_{ij} + A_{ji}$  minus  $A_{ji}$  which by definition of matrix addition is same as  $A_{ij} - A_{ji}$  right

so let me call this as one i will have to look at the transpose of  $A - A^T$   $A - A^T$  the whole transpose is the transpose of the matrix whose  $ij$ th entries are given by  $A_{ij} - A_{ji}$   $A_{ij} - A_{ji}$  whole transpose now let us try to apply the definition of that transpose this is same as  $A_{ji} - A_{ij}$  this is equal to minus of  $A_{ij} - A_{ji}$  which is same as minus of  $A_{ij} - A_{ji}$  which is equal to  $A_{ij} - A_{ji}$  and this matrix  $A_{ij} - A_{ji}$  this is exactly  $A - A^T$  suppose this follows from one thus  $A - A^T$  is a skew symmetric matrix now let us do an important theorem in this direction given any square matrix  $A$  it can be returned as a sum of a symmetric matrix and skew symmetric matrix right

so given a square matrix you can write it as a sum of a symmetric matrix under skew symmetric matrix the proof uses the fact that  $A + A^T$  is symmetric and  $A - A^T$  is skew symmetric well

so let  $A$  be any square matrix

so the claim is that  $A$  equal to  $B + C$  where  $B$  is a symmetric matrix and  $C$  is a skew symmetric matrix right i should write  $A$  as  $B + C$  where  $B$  is a symmetric matrix and  $C$  is a skew symmetric matrix fine how to do that

so let  $B$  equal to  $A + A^T$  and  $C$  equal to  $A - A^T$  then from what we have done previously it follows that  $B$  is a symmetric matrix and  $C$  is a skew symmetric matrix now the only thing that we will have to show is that  $B + C$  is  $A$  but this follows from the properties of the addition of matrices well let us try to prove it  $B + C$  equal to  $B + C$  is  $A + A^T + A - A^T$  well what i need is whole upon 2 that means i am just multiplying this matrix  $A + A^T$  by 2 and similarly  $A - A^T$  by 2 right

so all i need by two now by the properties of the scalar multiplication we know that  $\alpha(A + B)$  is  $\alpha A + \alpha B$

so this is same as a by two plus a transpose upon two plus a by two minus a transpose upon two

so this is same as a by two plus a transpose use the fact that it is associative

so the order is immaterial plus a by two minus a transpose upon two right

so by a by two the it means that you are multiplying the scalar half with the matrix a similarly for the other things and you have a by two a transpose by two and minus of a transpose by two

so therefore these two gets cancelled what you have is a by two plus a by two lets use the properties of the scalar multiplication this is half plus half times the matrix a which is one times a which is just a

so this is what we have

so thus b plus e is a or a equal to b plus c where b is a symmetric matrix and c is a skew symmetric matrix

so what have we done we have written a square matrix as a sum of a symmetric matrix and a skew symmetric matrix now let us do one more thing of what is known as the multiplication of matrices earlier we saw about multiplying a scalar with a matrix now let us do multiplication of matrices we are going to multiply two matrices

so here ah the order is very important one for multiplication right

so order

so order of

so certain things are very important

so let me just for state it let a be a matrix of order m by n and b b a matrix of order n by r right

so this n and this n

so the number of columns in matrix a should be same as the number of rows in the matrix b then the product of a and b denoted a b is obtained as follows

so as usual i will write here as c i j where i runs between one to m and j runs between one to n b i will write it as b i j where i runs between one to n and j runs between one to r and therefore the matrix a b

so let a b equal to c let me write it as c i j then

so the i j th entry that is the c i j is given as follows summation k running from one to n a i k b k j right

so summation one to n a i k b k j well

so now let us try to do one example i have a as one two three four b as five comma six here m is 2 n is 2 and r is 1 and order of a is two by two order of b is two by one these two matches and therefore a and b can be multiplied

therefore 1 2 3 4 multiplied with 5 and 6 this is going to give me the first entry summation one to two a i

so i the first entry is one k b k j it is only 1.

so i have only one column

so summation 1 to 2.

a two k b k one because i have only one column for b

so only this much is there fine let us try to expand things this is going to give me a one one b one one plus a one two b two second one a two one b one one plus a two two b two one what is a one one b one one a one one is one b one one is five plus a one two plus b two one six

so six into two twelve second one a two one three b one one which is five five into three fifteen plus a two two four into b two one which is six twenty four

so the final matrix that we have obtained is 17 and 39 this is the matrix that we have obtained let us do one more example let us do it for a one two three four this is your matrix a and your matrix b is one two three four five six let

us try to calculate right

so this is a two by two matrix you have two rows and two columns and this is a two by three matrix and if you notice these two sorry these two matches matches and therefore this is a these two are matrices which which are compatible for multiplication and therefore a b how do i multiply if you look at the previous one if you look at things what is happening you are multiplying the first row with the first this column and similarly this thing with this one

so first entry one with five and two and six that's what you have done right one with five and two ends two with six three with five and four with six let us do the same thing that is what it says

so one with one one times plus two with two which is four similarly one with three with plus two with four which is eight again one with five which is five plus two into six which is twelve second row three into one which is three plus four into two which is eight three into three which is nine plus four into four which is sixteen three into five which is fifteen plus four into six which is twenty four

so what is the final resulting matrix 5 11 17 11 25 and 39 this is the one that you have 5 11 17 11 25 and 39 let us try to go a bit further and try to do a bit difficult example let us look at two matrices a with entries 1 2 3 4 5 6 and 7 8 9 and matrix b with entries one two three four and five six if you look at the matrix a has got order three by three if you look at the matrix b it has got order three by two these two entries or these two things matches right this is of order m by n and this is the order n by k and therefore these two matches this three and this three matches and therefore these two matrices a and b are compatible for multiplication or they can be multiplied now let us try to multiply a b a and b and find out what a b is

so what we will have to do is multiply the first row with each of the columns till all the columns are exhausted one into one which is one two into three which is six plus six three into five which is fifteen next one one into two which is two plus two into four which is eight plus six into three which is eighteen second one four into one four plus five into three fifteen plus six into five which is thirty four into two eight plus five into four twenty plus six into six thirty six seven into one seven plus eight into three twenty four plus nine into five forty five last one seven into two fourteen plus eight into four thirty two plus nine into six fifty four let us write down the final one just by adding these six plus one seven seven plus fifteen twenty 2 plus 8 10 10 plus 18 28 4 plus 15 19 19 plus 30 49 20 plus twenty eight twenty eight plus thirty six sixty four seven plus twenty four which is thirty one thirty one plus forty five which is 76 14 plus 32 which is 46 46 plus 54 which is 100

so this is the final resulting matrix now let us look at one property regarding ah matrix multiplication and transpose for any two square matrices a and b of same order a b whole transpose is equal to b b transpose a transpose proof as usual lets write a as  $a_{ij}$  and b as  $b_{ij}$  right where in one less than or equal to i comma j less than or equal to n right that means you are assuming that a and b are of order n by n and this is possible because it is given that a and b are of same order and they are square matrices now let's c equal to a b let me write it as  $c_{ij}$  what does it mean by  $c_{ij}$  where the i j th entry of the matrix e is given by summation k running from one to n  $a_{ik} b_{kj}$  right this is what we have now let us try to calculate a b whole transpose what does this mean we are looking at transpose of c that means we are looking at the matrix  $e_{ij}$  and then taking the transpose of this which is equal to  $c_{ij}$  whole transpose  $c_{ji}$  which is going to give me the matrix with entry  $c_{ji}$  what are the  $c_{ji}$ 's i know what  $c_{ij}$  are

so let me just use it and then write down summation k running from 1 to n  $a_{jk}$

$b_{ki}$  right

so the matrix of  $a$   $b$  whole transpose have got entries summation  $k$  equal to 1 to  $n$   $a_{jk} b_{ki}$  these are the  $ij$ th entry once i have these now let us write try to calculate  $b$  transpose  $a$  transpose which is equal to look at the matrix with entries  $b_{ij}$  take its transpose similarly look at the matrix with entries  $a_{ij}$  take its transpose which means this will give me  $b_{ji}$  the other one will give me  $a_{ji}$

so i have two matrices with entries with  $ij$ th entry as  $b_{ji}$  and  $a_{ji}$  respectively i will have to multiply them

so the resulting matrix is going to be summation  $k$  running from 1 to  $n$  the  $i$  kth entry of this matrix  $b_{ki}$  and the other one i need the  $j$  kth entry which is  $a_{kj}$  right the first one  $b_{ki}$  represents the  $i$  kth entry of the first matrix and the second one  $a_{kj}$  represents the  $j$  kth entry of the second matrix which is equal to summation  $k$  running from 1 to  $n$  let me just rewrite this which is  $a_{kj} b_{ki}$  right you i have just used the fact that whether it is complex numbers or real numbers they are you know that they are committed multiplication is commutative i have just used this now if you look at what we calculated for  $a$   $b$  whole transpose these two matches and therefore this is same as  $a$   $b$  whole transpose thus  $a$   $b$  whole transpose equal to  $b$  transpose  $a$  transpose now let us do a simple example let us look at this matrix one two three four five six this is a matrix let me call it as  $a$  this is a matrix of order two by three and let me choose  $b$  as 1 2 3 4 5 six seven eight one two four five

so this the first one  $a$  is a matrix of order two by three and  $b$  its a matrix of order three by four and we know that the resulting matrix

so these two numbers matches therefore they are compatible for multiplying or they can be multiplied together therefore the resulting matrix is two by four matrix  $a$  two by four matrix on the other hand and therefore its a two by four matrix therefore let us try to calculate  $a$   $b$  1 plus 10 plus 3 2 plus 12 plus 6 3 plus 14 plus 12 4 plus 16 plus 15 next row 4 plus twenty five plus six eight plus thirty plus twelve twelve plus thirty five plus twenty four sixteen plus forty plus thirteen which is equal to eleven plus three fourteen twenty nine plus six thirty five fourteen plus six twenty thirty eight plus twelve fifty seventeen plus twelve twenty nine 47 plus 24 71 20 plus 15 35 fifty six plus thirty eighty six right we have this as  $a$   $b$  and therefore  $a$   $b$  whole transpose which is equal to i have

so this is a matrix of order two by four which means the transpose will be of order four by two fourteen 20 29 35 35 50 71 and 86 i have this now let us look at  $b$  transpose  $b$  transpose which is one two three four five six seven eight and one two four five we know that  $b$  is a matrix of order three by four and therefore  $b$  transpose this is a matrix of order four by three on the other hand  $a$  transpose which is given by one two three and four five six this is a matrix of order three by two and therefore

so these two matches they both are compatible for multiplying let us calculate  $b$  transpose  $a$  transpose 1 5 1 2 6 2 3 7 4 4 eight five into one four two five three six let us compute it which is given by 1 plus 10 plus 3 4 plus 20 plus 6 3 plus twelve plus six eight plus thirty plus twelve three plus fourteen plus twelve twelve plus thirty five plus twenty four four plus eight plus fifteen sixteen plus thirteen plus thirty sixteen plus forty right four plus four plus sixteen plus fifteen sixteen plus forty plus thirty which will finally give me fourteen fifteen plus six twenty one two plus twelve plus sorry this should be two which will give me twenty this is twenty nine thirty five thirty four plus twenty plus six sorry this should be twenty five

so they should be twenty five

so this will give me thirty five and then thirty eight this is fifty forty

seven plus twenty four seventy one seventy eighty six one can notice through this example that  $(A \cdot B)^T = B^T \cdot A^T$  i will stop here in the next class we will see some more properties of matrices and we will try to define the notion of what is known as invertibility of a matrix thank you

Prutor@iitk