

so welcome to this first session on problem solving for trigonometric and inverse trigonometric functions

so in the last lecture we finished our discussion on inverse trigonometric functions we solved a few problems also in the next two lectures will be basically solving some problems on both trigonometric and inverse trigonometric functions and thereafter will be starting a new topic on properties of triangles

so this is the first problem

so most of the problems that we will will be most of the problems that we will be discussing in today's lecture and the next lecture will be from je exam

so in this problem it is said that a , b and c are positive numbers and we are asked to find the value of the the sum of the tan inverses of three different values

so immediately it appears that we should use the tan inverse x plus tan inverse y formula

so we will start with adding the first two terms and before that ah what we see is that all these three values have $a + b + c$ as a common factor and then what we can also do is that we can actually multiply this first one by square root of a both in the numerator and denominator

so what we are going to get is

so if you multiply both the numerator and the denominator inside the square root with a a is going to come out and we are going to get a into square root of $a + b + c$ over $a + b + c$

so the first term is this quantity plus tan inverse of for the second term we are going to multiply this thing by square root of b

so both the numerator and the denominator

so we will end up getting the second term as tan inverse of b times square root of $a + b + c$ over $a + b + c$ and then we intend to use the tan inverse x plus tan inverse y formula and that is something that we have already covered in the previous one of the previous lectures

so so for the tan inverse x plus tan inverse y formula

so let me just recollect that

so this value will depend on the on the signs the sign of x and sign of y and also on the value of the product $x y$

so to be precise it is given by equals

so there are three cases

so the first case is if the product of x and y is less than one

so in that case this is equal to tan inverse of $x + y$ over $1 - xy$ and then the other cases are when x into y is greater than equal to one

so the other case is $x y$ is greater than equal to one and

so even for $x y$ greater than equal to one there are two sub cases

so when $x y$ is greater than equal to one either both x and y are positive or they are both negative

so so this is the

so this is the second case

so if if if we have this case then the value is given by

so this expression is still going to remain

so it will be π plus tan inverse of $x + y$ over $1 - xy$ and the third case is when again when x the product $x y$ is greater than equal to one but both x and y are negative and in that case the expression is minus π plus tan inverse of $x + y$ over $1 - xy$

so now we have to in order to use this formula we have with this

so in this case we would like to in our case now we would like to evaluate this expression

so for us let us say this is x and this is y and then we have to first look at the product of x and y

so the product of x and y is $a + b + c$ over this will be over c because we will get $a \times b$ and then in the denominator we have $a + b + c$

so the $a + b$ is going to get cancelled and what remains in the denominator is only c and this obviously is strictly greater than one because it is given that $a + b$ and c are all positive and therefore for this case

so the

so this is the condition which gets satisfied for us now because if you see this is positive because $a + b$ and $c + 1$ are positive this is also positive because all $a + b$ and c are positive plus $x + y$ is strictly greater than one

so this second case is the one which we qualify for and therefore \tan^{-1} of a into square root of $a + b + c$ over $a + b + c$ plus \tan^{-1} of b into square root of \tan^{-1}

so \tan^{-1} of a into square root of $a + b + c$ over $a + b + c$ plus \tan^{-1} of b into square root of $a + b + c$ over $a + b + c$ is going to be equal to

so we take this second expression π plus \tan^{-1} of $x + y$

so $x + y$ is going to be this plus this which can be written as $a + b$ into square root of $a + b + c$ over $a + b + c$ divided by one minus $x + y$ $x + y$ is or we have already seen that it is equal to $a + b + c$ over c and further simplification will give us π plus \tan^{-1} of $a + b + c$ over c

so this denominator is going to become

so after simplification this is going to become minus c into square root of $a + b + c$ over $a + b + c$ now we know that the \tan^{-1} function is an odd function and therefore \tan^{-1} of x \tan^{-1} of $-x$ is equal to

so for any x \tan^{-1} of $-x$ is equal to minus of \tan^{-1} of x

so this is the result we are going to use here and then we get this to be equal to π minus \tan^{-1} of c into square root of $a + b + c$ over $a + b + c$ and therefore if we take this sum on this side we exactly get these three terms

because this last term if you see this last term is precisely equal to this term and therefore when we take this term from the right hand side to the left hand side what we will end up getting is that this entire sum will be equal to π

so that finishes the first problem

so here is the next problem

so in this problem for all $\cos x$ less than equal to one we are asked to find the smallest and the greatest value of a^h this quantity which is $\sin^{-1} x$ to the power four plus a^h plus $\cos^{-1} x$ to the power four now we know that $\sin^{-1} x$ has a range which is from minus $\pi/2$ to plus $\pi/2$ $\cos^{-1} x$ belongs to the interval zero to π but we also know the result that for any x such that $\cos x$ is less than equal to one we know that $\sin^{-1} x$ plus $\cos^{-1} x$ is $\pi/2$

so this is true and though we have powers of four here we would be interested in using a^h this identity somehow

so if you want to use that identity then we can write this expression as equal to $\sin^{-1} x$ to the power 4 plus $\pi/2$ minus $\sin^{-1} x$ to the power four and just to simplify this let us say that a^h θ is equal to $\sin^{-1} x$ and of course θ has to belong to the interval minus $\pi/2$ to plus $\pi/2$ which is the range set of \sin^{-1} and then this entire expression becomes in terms of θ that is equal to θ to the power 4 plus $\pi/2$ minus θ to the power 4

so θ to the power four plus $\pi/2$ minus θ to the power four

so what we are asked to do is to find the the largest and the smallest values

so let me say that this is f of θ

so immediately what comes to mind is to take the first derivative of $f(\theta)$
so the first derivative of $f(\theta)$ is going to be $4\theta^3 - 4(\pi/2 - \theta)^3$
into $\pi/2 - \theta$ by two minus θ^3

so this is the first derivative and to find the extremum points we have to
equate this to zero

so then the equation that we get is $\theta^3 - (\pi/2 - \theta)^3 = 0$ or in other words θ^3 is equal to $(\pi/2 - \theta)^3$
now since both θ and $\pi/2 - \theta$ are real

so the the only solution to this is only a real solution to this particular
equation is that θ is equal to $\pi/2 - \theta$ which implies that
there is an extremum point at $\theta = \pi/4$ and that because $\sin x$
so the corresponding value of x is going to be $\sin(\pi/4)$ which is $1/\sqrt{2}$
over root two

so now but we have to see whether this is a maxima or a minima point whether it
is a maximum or a minimum for which we have to take the second derivative of $f(\theta)$
this function $f(\theta)$ with respect to θ

so this second derivative of

so the $f(\theta)$ there is one subtle point also here that we must realize is that we
are asked to find the maximum minima of this particular function but we are not
asked to find the value of x for which the maximum or minimize obtained we are
just asked to find the the value of the function the maximum or the minimum
value of the function as it is

so that value is going to be the same as the maximum or the minimum value of
this function

so that is why we have made this substitution and will only focus our attention
on this particular function now

so now this second derivative will be equal to $12\theta^2 + 12(\pi/2 - \theta)^2$ which as you can see is greater than
zero and therefore it implies it is implied that $\theta = \pi/4$ is a
minimum point

so its a minimum of for $f(\theta)$ and the interesting thing is that it does lie
so this $\theta = \pi/4$ does lie in the interval $-\pi/2$ to
plus $\pi/2$

so we are certainly sure that the therefore the minimum value of $f(\theta)$ for
 θ belonging to $-\pi/2$ to plus $\pi/2$ is

so if you remember $f(\theta)$ was this function and

so essentially this and this are equal at $\theta = \pi/4$

so the value automatically becomes $2 \times (\pi/4)^4$ to the power of four
which is π^4 over four

so 4 to the power 4 is 256

so it is 128 .

so this is the minimum value of this

so the minimum value of $\sin^{-1} x$ to the power four plus $\cos^{-1} x$ to
the power four is π^4 over one twenty eight now the the tricky
part is to actually find the maximum value and we know that there is no maxima
of this function but then since the domain of this function is limited to $-\pi/2$
to plus $\pi/2$ which is a finite interval finite length interval

so there has to be a maxima somewhere as long as θ is restricted to this
finite interval

so for that we have to actually ah try to look at the first derivative and see
what is happening

so if we see that the first derivative was $4\theta^3 - 4(\pi/2 - \theta)^3$ now it is clear from here that whenever

so since see the theta the value of theta is between minus pi by two to plus pi by two

so we need to examine the value of the first derivative or rather plot a graph for $f'(\theta)$ with respect to theta we know that there is only one place where it is equal to zero but what about the value of the first derivative at other values of theta in this interval

so ah it is clear that when theta is greater than pi by 2 minus theta let us consider this region of theta

so when this is true that implies that

so when this is true then well it actually depends on the values taken by whether it is positive or negative

so a little graphical plot would actually help us here

so here we have theta on the horizontal axis and on the vertical axis we are going to plot $\theta^3 - \pi/2 - \theta^3$ and we have theta let us say between minus pi by 2 and plus pi by 2

so let us say this is $\pi/4$

so this is $\pi/4$ this is 0 this is minus pi over four and this is minus pi over two

so we know that this value is exactly equal to zero at theta equal to pi by four

so now let see what happens when when theta is greater than pi by four and of course less than equal to pi by 2 which is this region

so when theta is between these two values then both theta as well as pi by 2 minus theta are positive and also in this region theta is greater than pi by 2 minus theta and therefore this first derivative has to be the first derivative has to be positive

so in this region

so in this region it is clear that $f'(\theta)$ is greater than equal to zero

so which means that if but but because it is an cubic

so it will look something like ah and since it is always positive

so might take some value like this

so this is r

so this is let us say this is the curve for $f'(\theta)$ in this interval and then let us then look at ah the other interval which which is for let us say theta is less than equal to pi over four but theta is positive when theta is satisfying this condition what we have is that theta will be less than pi by two minus theta

so when theta is in this region this is true and therefore now since both theta is also positive and pi by 2 minus theta is also positive and theta is less than pi by 2 minus theta for when we are in this region

so what will happen is that this the value of this first derivative is going to be negative is going to be negative because theta is less than pi by 2 minus theta and both theta and pi by 2 minus theta are positive

so therefore it will be negative which means that $f'(\theta)$ will be let us say somewhere here something like that

so we have $f'(\theta)$ is negative in this interval and then similarly we have to consider the other we have to consider negative values of theta

so when theta is less than equal to zero

so when theta is negative what we see is that ah theta cube is going to be negative

so when theta is negative theta cube is also negative and what we see is that pi by 2 minus theta will be a positive value and therefore pi by 2 minus theta q will be positive but because there is a negative sign here minus of pi by two minus theta cube is also going to be negative and therefore when theta is less

than equal to zero the first derivative $f'(\theta)$ is also again going to be negative

so if i had to plot it further

so let me replot it essentially this graph this curve would be still negative like something like this

so the the graph is not exact what is more important here is whether $f'(\theta)$ is positive or negative

so just to summarize the discussion on the previous slide what we have found out is that when θ is greater than equal to $\frac{\pi}{4}$ and less than equal to $\frac{\pi}{2}$ we have the first derivative as positive and when θ is less than equal to $\frac{\pi}{4}$ and greater than equal to $-\frac{\pi}{2}$ then the first derivative is negative therefore from this what we actually if we were to from this and of course we know that at $\theta = \frac{\pi}{4}$ the derivative is equal to zero

so we can approximately plot the graph of $f(\theta)$

so we have θ here

so we have $\frac{\pi}{2}$ minus $\frac{\pi}{2}$ and

so this is $\frac{\pi}{4}$ this is $-\frac{\pi}{4}$.

so what is going to happen is that the minimum value as we have seen is π to the power 4 by 128 eight

so lets say this value here is π to the power four by one twenty eight and that minimum value is attained at $\frac{\pi}{4}$ now we know that in this interval $f'(\theta)$ is positive and therefore $f(\theta)$ will increase monotonically in this interval

so it might go something like that and then in this interval from $-\frac{\pi}{2}$ to $\frac{\pi}{4}$ we know that it is negative which means that it would be decreasing starting from here the value would be decreasing like that please note that these are not exact values these are just indicative values ok just to explain you that the function is first decreasing from and it is this decreases monotonic because in this entire range the derivative the first derivative is negative

so this decreases monotonic

so it first decreases monotonically from $-\frac{\pi}{2}$ to $\frac{\pi}{4}$ and then from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ it increases monotonically

so the graph is expected to be something like this please note that this is not an exact graph and also in the previous slide please note that this is not an exact graph it was just for the sake of illustration and anyways what is most important for us in this problem is to know the regions where the derivative is positive and where it is negative

so from this it is clear that the minimum is at $\frac{\pi}{4}$ but then we see that clearly see that the maximum of $f(\theta)$ in this interval will be either this value or this value and therefore we need to calculate these two values and compare them

so the

so essentially what we have to do now is to find the value of $f(\theta)$ which is $\theta^4 + \pi^2 - \theta^4$ at both the end points

so the value at $\theta = -\frac{\pi}{2}$ will be $\pi^4 + 16$ and this will be π^4 the value at $\theta = \frac{\pi}{2}$ will be only π^4

so obviously this is the larger value and hence the the maximum value of $\sin^{-1} x$ to the power four plus $\cos^{-1} x$ to the power four is equal to $\frac{\pi^4}{16} + \pi^4$ well that is actually $\frac{17\pi^4}{16}$

so this is the maximum value and the minimum value is π to the power four over one twenty eight

so this finishes the solution to the second problem another very interesting problem is asking us to find the number of solutions to this particular trigonometric equation but only when x is constrained to lie in the interval $[-2\pi, 2\pi]$

so for such many times we have to use graphical techniques because it is not possible to exactly solve and find the points where this left hand side and right hand side are equal

so this is one such example

so let us

so the way we are going to solve this problem is that we are going to actually understand and compute this particular function

so we will start with $\sin^{-1}(\sin x)$ and see how it looks when x is constrained to this interval here and that is not very difficult because

so let us say that $\sin^{-1}(\sin x)$ is equal to y

so of course this value of y has to belong to the range set of the sine inverse function which is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ but we need to express this y in terms of x that is what our ultimate goal is going to be

so from here if we take the if we apply the sine function on both the sides then we get $\sin x$ is equal to $\sin y$

so now we divide this into several regions

so the first region is when x is let's say between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ by two

so when x is in this interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we know that if we restrict to this interval then $\sin x$ is a monotonic function and therefore if $\sin x$ is equal to $\sin y$ then it must be true that y is equal to x because remember that y already belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ two

so if x is in this interval then $\sin^{-1}(\sin x)$ is equal to x and this we have already seen several times in previous lectures the next interval is x is greater than equal to $\frac{\pi}{2}$ and it is less than equal to $\frac{3\pi}{2}$.

so obviously when x is in this interval $\sin^{-1}(\sin x)$ cannot be equal to x because $\sin^{-1}(\sin x)$ has to belong to this interval whereas x is not belonging to that interval

so but for this interval what we see is that if x belongs to this interval then $\pi - x$ is going to belong to the range set of sine inverse which is this set and further we know that $\sin(\pi - x)$ is equal to $\sin x$ which is already equal to $\sin y$

so what we have here is that $\sin(\pi - x)$ is equal to $\sin y$ and $\pi - x$

so y is already belonging to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ $\pi - x$ is also belonging to this interval and therefore it must be true that $\pi - x$ is equal to y

so what implies is implied from here is that $\pi - x$ is equal to y which implies that

so for the second case

so for the second case which is x greater than equal to $\frac{\pi}{2}$ and less than equal to $\frac{3\pi}{2}$ what we have is that $\pi - x$ is y which is $\sin^{-1}(\sin x)$ therefore $\sin^{-1}(\sin x)$ is $\pi - x$ if x belongs to this interval similarly if x is greater than equal to $\frac{3\pi}{2}$ less than $\frac{5\pi}{2}$ then again in a similar manner since we know that if x is in this interval then $x - 2\pi$ is going to belong to again the range set of sine inverse and because we are just shifting x by 2π it is clear

that sine of $x - 2\pi$ is equal to sine x which is sine y

so here what we have is sine of $x - 2\pi$ sine y and both $x - 2\pi$ and y belong to the range set of sine inverse which is $-\pi/2$ to $\pi/2$ then therefore it must be true that y is equal to $x - 2\pi$ and a similar kind of argument can be done for the negative x and after all this effort what we are going to get is that

so for x between $-\pi/2$ and $\pi/2$ this is how sine inverse of sine x is going to look and then we plot this exactly on

so here we plot

so we have three different curves in the same plot

so the horizontal axis is representing x and we have to limit ourselves to the interval $-\pi/2$ to $\pi/2$ and that is because it was asked in the question that how many solutions are there to this equation for x restricted to this interval

so the blue curve here is for sine inverse of sine x

so that is blue curve here and then \cos of sine inverse sine x has been plotted with a dotted red line shown in red and that is very easy to plot because when if whenever the blue line is on the positive upper half the red line is going to be exactly the same but when the blue line is on the negative half then the red line is just going to be a mirror image about the x axis

so that is how we get \cos of sine inverse sine x and then since we had to find the number of solutions to or number of distinct different points x such that \cos of sine inverse sine x is equal to $\cos x$

so we have to also plot a graph for $\cos x$

so in black

so this black curve here is for \cos of x and it is very clear that

so now what we have to ultimately find out is the places where the red dotted curve and the blue and the black curve are intersecting

so the first place is over here and then the second place is here and then immediately we get another point over here and then we get the last point over here

so we do not go beyond $\pi/2$ or $-\pi/2$ because we have to restrict ourselves to $-\pi/2$ to $\pi/2$

so we see that there are four distinct solutions where both these curves will meet and therefore the number of solutions to the equation \cos of sine inverse sine x equal to $\cos x$ is four

so that solves the third problem also

so what we see is that many times we have to take use of graphical methods also

so that is one take from this particular question

so here is another interesting problem and this is also a problem from one of the previous exams

so it says that let S be the set

so we are essentially restricting our x to the open interval $-\pi/2$ to $\pi/2$ and the x is not allowed to take the value 0 and $\pi/2$ and $-\pi/2$ and then what we have here is a trigonometric equation and we are asked to find the sum of all the distinct solutions to this trigonometric equations

so obviously we have to first solve this trigonometric equation and then take the sum of all the distinct solutions

so the word distinct is also very important because sometimes we might get double roots and

so basically two values of the same x

so starting we basically have that $\sqrt{3} \sec x + \csc x + 2 \tan x$

so this is the trigonometric equation and this of course is $\sqrt{3} \sec x$

plus 1 over sine x plus 2 times sine x by cos x minus cos x by sine x equal to zero now since it was mentioned that x cannot be plus minus pi by two and also not zero and x can also not be equal to plus or minus pi

so since x

so since x cannot be 0 and neither can it be plus minus pi

so it is clear that for all x belonging to S sign x is not equal to zero similarly since x is not equal to plus minus pi by two for all x belonging to the set S cos x is also not equal to zero and hence their product is also not equal to zero

so we know that if x belongs to S for all x belongs to S then sin x times cos x is not zero and therefore now if we multiply both the sides of this equation with sine x cos x what we end up getting is that root three sine x plus cos x plus two times sine square x minus cos square x equals zero and a little bit of rearranging these terms is going to give us root 3 by 2 sine x plus half cos x is equal to

so this we take it to the right hand side it becomes cos square x minus sin square x and we know that cos square x minus sin square x is cos of two x and this appears to be of the form cos a cos b plus sign a sin b

so this we can write as cos x into cos pi by three plus sine x into sine pi by three equals cos of two x but cos a cos b plus sin a sin b is cos a minus b and hence what we get is cos of x minus pi over three is equal to cos of two x

so x will satisfy this equation if and only if it satisfies this equation as long as x is restricted to the interval S taking this forward and here we are going to essentially use something that we have already seen when we were discussing trigonometric equations

so if you remember we said that if cos x is equal to cos y then it must be true that x is equal to y

so sorry x is equal to two n pi plus minus y for some integer n

so this is from one of our previous lectures and what we have here is that cos of x minus pi by three is equal to cos of two x

so essentially what it means is that this two x must be equal to i mean we can write it either way

so we can even write it like this

so this is what must be true

so essentially ah what we are doing here is ah we are using this equation and we are taking this as our y in this equation and this is we are taking it as this x

so if we use if we use this identity then this is what we end up getting that if cos of x minus pi by three is equal to cos of two x then it must be true that two x is equal to two n pi plus minus x minus pi by three but here n should be an integer

so if we start first with the plus sign here what we get is two x must be two n pi plus x minus pi over three which implies that x must be of the form two n pi minus pi over three but remember that ah we are only allowed the value of x to x must belong only to the interval minus pi to plus pi and therefore all values of n might not be permissible

so if we take n equal to 0 then the value of x that we get is minus pi by 3 which definitely belongs to the interval minus pi to plus pi but if we take any other value of n any other integer value like 1 then the value of x that we get with n equal to 1 is 2 pi minus pi by 3 and this value is definitely not in the interval

so this does not belong to the interval minus pi to plus pi

so this is not a valid solution for us and the same thing will happen if we take any other integer multiple of any other integer multiple which is not zero

so with the plus sign here the only solution that we get is x equal to $-\pi/3$ now let us see what we get with the negative sign here

so that equation is $2x$ must be equal to $2n\pi - x - \pi/3$ and that can then be written as $3x$ is equal to $2n\pi + \pi/3$ or x is equal to $2n\pi/3 + \pi/9$ again starting with n equal to zero we get x equal to $\pi/9$ which of course belongs to the interval $-\pi/3$ to $\pi/3$ with n equal to one we get x equal to $2\pi/3 + \pi/9$

so so this is also a valid solution for us because this also belongs to the interval $-\pi/3$ to $\pi/3$ but we if we take larger values of n like n equal to two and

so on then the value that we get will not belong to $-\pi/3$ to $\pi/3$ and therefore those are not valid solutions for us on the negative side if we take n equal to -1 then we get x equal to $-2\pi/3 + \pi/9$ and this value of course belongs to the interval $-\pi/3$ to $\pi/3$

so $-\pi/3$ to $\pi/3$ but if we take n equal to -2 then that value will not belong to the interval $-\pi/3$ to $\pi/3$

so so that solution is not alone

so with the negative sign here we get 3 valid solutions and with the positive sign with the positive sign we only got one solution which was x equal to $-\pi/3$

so that is the fourth solution

so totally we get four different value solutions of x for that for this trigonometric equation and then we were if you remember we were actually asked to find the sum of all the distinct solutions

so as we can see all of these solutions are actually distinct

so none of them is equal to the other and when we take their sum what we get is so of course this and this will cancel and when you add this and this and this will get $\pi/3$ but that when added to this will give us zero

so it turns out that the sum of all these four distinct values is in fact equal to zero

so the final answer is that the sum of all the distinct solutions to this equation where the solutions belong to the set S is equal to zero if we go back a few steps back then we started with \cos of $2x$ equal to \cos of $x - \pi/3$ and we said that the general solution to this equation is of the form that $2x$ must be equal to $2n\pi + x - \pi/3$ now some student ah might wonder that since \cos is an even function \cos of $x - \pi/3$ is x actually also equal to \cos of $\pi/3 - x$

so but if we if we start with \cos of $2x$ equal to \cos of $\pi/3 - x$ and then solve it exactly in the manner we did are we going to get the same set of solutions and the answer to that that question is of course yes and that is because if if if we lets say start with this equation

so instead of $x - \pi/3$ we have $\pi/3 - x$

so again the the same because the general solution to this equation will also be of the form to $n\pi + \pi/3 - x$ but if we see this the important thing here to see is this plus minus

so because the we have a plus here

so for with the plus what we will get is $2n\pi + x - \pi/3$ when we take the minus we take we get $2n\pi - x - \pi/3$ but which is actually

so when we take the minus sign here that is the expression is exactly equal to what we get in this case with the plus sign and similarly what we get here with the minus sign is exactly equal to what we get here with the plus sign

so that is why both these things are going to be same they are going to give us the same solution

so with that will end this lecture and in the next lecture we will do some more similar problems thank you you

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