

welcome to the third lecture on inverse trigonometric functions in the last lecture we saw some of the identities and relations between some of the inverse trigonometric functions in this lecture also will continue to do

so and hopefully finish it and in the subsequent lectures we are going to take up problems before we start deriving some new relations here is a general problem that we are going to face throughout this lecture and also may be while problem solving solving problems related to inverse trigonometric functions

so for example here let us say that we are asked to find a value of  $x$  between minus one and zero such that  $\sin^{-1} x = \frac{\pi}{6}$

so somebody could try this route

so somebody could write say that ok half is equal to  $\sin$  of 30 degrees which is  $\frac{\pi}{6}$  and then equate  $\cos^{-1} x$  with  $\frac{\pi}{6}$ .

so you could write  $\cos^{-1} x$  is therefore  $\frac{\pi}{6}$  and then taking  $\cos$  on both the sides here you would get  $x = \cos\left(\frac{\pi}{6}\right)$  which is equal to  $\frac{\sqrt{3}}{2}$  the problem is that  $\frac{\sqrt{3}}{2}$  does not belong to the interval  $[-1, 0]$

so so what do we do in such a case the thing to be realized here is that if  $x$  is between minus one and zero then from the graph of the  $\cos^{-1}$  function you can see that  $\cos^{-1} x$  would belong to the interval  $[\frac{\pi}{2}, \pi]$  but then the problem is that  $\sin^{-1} \frac{1}{2}$

so if you remember we said that  $\cos^{-1} x$  is equal to

$\frac{\pi}{6}$  is  $\sin^{-1} \frac{1}{2}$

so the problem with equating this  $\cos^{-1} x$  to  $\sin^{-1} \frac{1}{2}$  is that  $\sin^{-1} \frac{1}{2}$  will always belong to the interval  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  and that is because the range set of  $\sin^{-1}$  is  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  whereas if  $x$  is in the interval  $[-1, 0]$  then  $\cos^{-1} x$  will belong to another interval  $[\frac{\pi}{2}, \pi]$  and we see that this interval and the range set of  $\sin^{-1}$  are mutually exclusive

so the intersection of these two is the null set and therefore we end up with the problem that we got a value of  $x$  which does not lie in this interval

so we will be facing a similar this is the basic pattern of the problem that we might face

so the solution out of this is that again going back to the example

so obviously if  $x$  is between minus one and zero then  $\cos^{-1} x$  which I denote by  $\theta$  will definitely lie between  $\frac{\pi}{2}$  and  $\pi$  and then instead of going and straight away saying that  $\cos^{-1} x = \sin^{-1} \frac{1}{2}$  which is not possible because  $\sin^{-1} \frac{1}{2}$  will not belong to this set

so  $\cos^{-1} x$  cannot be equal to  $\sin^{-1} \frac{1}{2}$  but then we see that  $\sin(\pi - \theta)$  is the same as  $\sin \theta$  and therefore we can

so the the reason why we took this choice of  $\pi - \theta$  is because if  $\theta$  equal to  $\cos^{-1} x$  belongs to this interval then we will see that  $\pi - \theta$  if  $\theta$  belongs to this interval then  $\pi - \theta$  belongs to the interval  $[0, \frac{\pi}{2}]$

so and this interval  $[0, \frac{\pi}{2}]$  now that we get belongs to the range set of  $\sin^{-1}$

so this is the basic trick that has to be done and of course ah we know that

so this  $\sin \theta$  is actually  $\sin(\cos^{-1} x)$  and that is given to be in the problem this value is given to be half

so what we have is  $\sin(\pi - \theta) = \frac{1}{2}$  and then obviously we can say that  $\pi - \theta = \sin^{-1} \frac{1}{2}$  the reason why

so from here we can conclude this line only because this  $\pi - \theta$  belongs to the range set of  $\sin^{-1}$

so  $\pi - \theta$  belongs to  $[0, \frac{\pi}{2}]$  which is a subset of the range

set of sine inverse the range set of sine inverse was minus pi by two to pi by two

so because of the fact that pi minus theta belongs to the range set of sine inverse from here we can get this and this basically then means is that since theta is cos inverse of x

so what we get is pi minus cos inverse x is sine inverse of half of course sine inverse of half equals pi by 6 and therefore we get cos inverse x equals pi minus pi by 6 which is 5 pi over 6 and therefore x is equal to now if we take the cos on both the left and the right hand side we get x equal to cos of five pi by six which is going to be equal to minus cos pi over six which is equal to minus square root of three over two

so finally we get x equal to minus square root of three over two which belongs to this interval minus one to zero

so this is the correct solution

so the essential the the main point here was that was that if we directly ah try to substitute cos inverse x as equal to sin inverse of half then we get into a problem because the ah sine inverse of half belongs to the range of sine inverse which is minus pi by 2 plus pi by 2 whereas for x negative cos inverse of x does not belong to the range

so so essentially if we if we if we do if we equate it like this then for sure the value of x that you get here is not going to lie in the interval minus one to zero

so we can formalize this thing in the next slide i have tried to formalize this

so the essential problem is that suppose that f is a trigonometric function of the form it could be any of these six trigonometric functions that we have defined

so in general the trigonometric function has a domain a and range b and then we went on to defining the inverse of each of these six trigonometric functions

so the inverse function f inverse obviously the domain set is going to be the same as the range set of the function f which is b

so the domain of f inverse is the set b and the range set will be another set c which of course has to be a subset of the set a which is the domain of the function f

so the general problem is that suppose if we are said that we have to solve this equation where we have to find theta

so the value of x is given and we have to find theta

so that f of theta equals x now we are already said that theta belongs to the range set of f inverse then the solution is very easy its the solution is simply theta equal to f inverse of x because f inverse of x will belong to the range set of f inverse which is c therefore theta will all will definitely belong to

so if if if we say that theta has to belong to c then this solution satisfies this constraint that theta has to belong to the range of range set of f inverse but if lets say that we have the same problem f theta equal to x we have to find theta but it said it is said that we need to find the value of theta which does not belong to the range set of f inverse then the question is how do we find theta now obviously theta cannot be equal to

so in this case in this case where we are said that theta does not belong to the range set of f inverse it is it is clear that theta is not equal to f inverse of x because f inverse of x will always belong to the set c whereas the clear it is clearly mentioned here that theta must not belong to the set c

so how do we find theta in such a case

so i have tried to solve this problem in general for the sine function

so for the sine function the domain and the range is as written here and for the sine inverse function the domain is minus one to one the range is minus pi

by two to plus pi by two and let us say that we have to solve this equation here so we have to find the theta such that sin theta equal to x x of course belongs to minus one to plus one

so we have to find theta but what we are given is that theta belongs to this range

so where m is some integer

so this is what we are asked to find we are asked to find a theta which belongs to this range  $m\pi - \pi/2$  to  $m\pi + \pi/2$  to this interval such a way that sin theta is equal to x

so how do we find it

so this is not very ah tough of course we need to consider even an odd m

so so let us say that if

so we start with writing this equation sine theta equals x and we are said that theta belongs to this interval but the problem is that of course if if we take the special case of m equal to zero then what the this basically this constraint over here translates to theta belonging to minus pi by two to plus pi by two and this is the range set of sine inverse and therefore for the special case of m equal to zero the solution is very easy theta equals to sine inverse x

so this is the solution for m equal to zero because for m equal to zero this set where we are constraining theta to line will essentially become the range set of sine inverse but what if m is not equal to zero

so in the case where m equal to naught m naught equal to zero will have to do something more

so we again start with sin theta equal to x and we are said that theta belongs to  $m\pi - \pi/2$  to  $m\pi + \pi/2$  the closed interval but what we see is that theta minus m pi if theta belongs to this interval then theta minus m pi belongs to the interval minus pi by two to plus pi by two which and this interval is basically the range set of sine inverse now let us see what is the value of sine of theta minus m pi and we will see that this is equal to sine a cos b minus cos a sine b but sine of an integer multiple of pi is equal to zero

so this is what we end up getting and this cos of m pi can be written as minus one to the power of m

so this is equal to finally equal to minus one to the power m times sin theta which is actually good news for us because we can also write it as minus one to the power of m times x because sin theta is equal to x

so then we can divide this into two cases one is when m is even

so when m is even this equation essentially becomes sine of theta minus m pi equals x and then if you see here this theta minus m pi obviously belongs to the range set of sine inverse and therefore from this equation we can directly write that theta minus m pi equals sine inverse of x from where we can conclude that theta is equal to m pi plus sine inverse of x

so in the case where m is even this is the solution

so and then for the case of odd m it is going to be similar for odd m what we are going to see is that sine of theta minus m pi equal to minus one to the power m times x is simply minus of x and we already know that this theta minus m pi belongs to the range set of sine inverse and therefore from this statement we can directly say that theta minus m pi is equal to sine inverse of minus x but sine inverse is an odd function and therefore this is equal to minus of sine inverse x from where we finally conclude that when m is odd theta equals to m pi minus sine inverse x

so this whole thing can be summarized as in the following slide here

so the basic problem was to find a theta in this interval  $m\pi - \pi/2$  to  $m\pi + \pi/2$  where m is some integer and we have to find this theta in this interval and theta should be such that sin theta is equal to some value of

some value  $x$

so the general solution for that is if  $m$  is even then  $\theta$  equals  $m\pi$  plus  $\sin^{-1} x$  if  $m$  is odd then it is  $m\pi$  minus  $\sin^{-1} x$  similar kind of equations can be derived for the other trigonometric functions also we now move over to the problem or to the identity which we had where we had stopped in the previous lecture

so if you remember we were trying to see if  $\tan^{-1} x$  plus  $\tan^{-1} y$  can be written as  $\tan^{-1}$  of something

so here is a quick derivation

so  $\tan$  of we know that  $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$  now in this formula here if I substitute  $a$  equal to  $\tan^{-1} x$  and  $b$  equal to  $\tan^{-1} y$  then this is what I get

so  $\tan$  of this is  $x$  and this is  $y$

so  $\tan(a + b)$  is  $\frac{\tan a + \tan b}{1 - \tan a \tan b}$  and of course  $\tan(\tan^{-1} x)$  is  $x$  and  $\tan(\tan^{-1} y)$  equals  $y$  and therefore this is what we get

so ultimately what we get is  $\tan$  of the sum of  $\tan^{-1} x$  and  $\tan^{-1} y$  is simply equal to  $\frac{x + y}{1 - xy}$

so we have this statement here but does this necessarily imply that  $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$  is this always true the answer is no it depends on the values of  $x$  and  $y$  because we know that the range set of the  $\tan^{-1}$  function the range set is  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$

so therefore if  $x$  and  $y$  are such that  $\tan^{-1} x + \tan^{-1} y$  if it does not belong to the range set of  $\tan^{-1}$  then we cannot have this equality here

so that is the problem

so in the next few slides I will derive the following result

so here is a nice table

so if we will show that if  $xy < 1$  then it is true that  $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$  but

so this is equal to this only if  $xy < 1$  but for the other cases

so the other cases if both  $x$  and  $y$  are positive and  $xy > 1$  in that case we will have to add a  $\pi$  to this expression

so this is what we get on the other hand if both  $x$  and  $y$  are negative but the product is still greater than one we add a minus  $\pi$

so to show these three conditions we will need a small result which is something that we have already seen in the previous lecture but I still thought of writing it down here

so from the previous lectures we have seen both these properties

so one was that for any  $x$  real this sum of  $\tan^{-1} x$  and  $\cot^{-1} x$  is  $\frac{\pi}{2}$  always and also that if  $x$  is positive then  $\tan^{-1} \frac{1}{x}$  is the same as  $\cot^{-1} x$

so now if we try to combine both of these statements both the statements

so then we can write that for  $x$  non-negative we just replace this  $\cot^{-1} x$  over here is replaced by  $\tan^{-1} \frac{1}{x}$

so then we get this particular equation which is that for all  $x$  non-negative  $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$  and then if you multiply this whole thing by a minus sign we get  $-\tan^{-1} x - \tan^{-1} \frac{1}{x} = -\frac{\pi}{2}$  for all  $x > 0$  but since the  $\tan^{-1}$  function is an odd function we can write  $-\tan^{-1} x = \tan^{-1}(-x)$  and the same for  $-\tan^{-1} \frac{1}{x} = \tan^{-1}(-\frac{1}{x})$

so then this is what we end up getting but the condition  $x > 0$  can also be written as  $x < 0$ .

so now we see that in this equation we have  $\tan^{-1} x$  everywhere

so we say if we use another variable  $y$  equal to  $\tan^{-1} x$  then the same thing can be written as  $\tan^{-1} y + \tan^{-1} \frac{1}{y} = \frac{\pi}{2}$  for all negative  $y$

so we will probably just be using this and these two equations which look very similar except that we have  $\frac{\pi}{2} - \tan^{-1} x$  here

so let us take the case where  $x > 0$  when the product of  $x$  times  $y$  is equal to 1

so when  $x > 0$  but both  $x$  and  $y$  are positive

so since  $x$  and  $y$  are both positive  $\tan^{-1} x$  will belong to the interval  $(0, \frac{\pi}{2})$  and that is clear from this graph of  $\tan^{-1} x$

so if we just look at the graph for positive values of  $x$

so for these values of  $x$  we see that the  $\tan^{-1} x$  is taking values on the positive side of the vertical axis

so we

so that is why when  $\tan^{-1} x$  is greater than zero then  $\tan^{-1} x$  will belong to this interval from zero to  $\frac{\pi}{2}$  and the same is true for  $\tan^{-1} y$

so  $\tan^{-1} y$  because  $y$  is positive  $\tan^{-1} y$  will also belong to zero to  $\frac{\pi}{2}$  but we also have this third condition that  $x > 0$

so that basically means that  $y < \frac{1}{x}$  but both  $x$  and  $y$  are positive because  $x$  and  $y$  are positive

so we have this thing here

so we will try to focus only on the positive  $x$  axis of this curve now you see that on the positive side of the horizontal axis the graph of  $\tan^{-1} x$  is monotonically increasing and therefore from here we can say that  $\tan^{-1} y$  is less than  $\tan^{-1} \frac{1}{x}$  now we add  $\tan^{-1} x$  on both the sides

so what we get is and of course

so then what we get is of course we have a greater than zero sign over here and we also know that  $\tan^{-1} x$  is greater than zero because both  $x$  and  $y$  are positive now to this equation here we will add  $\tan^{-1} x$  everywhere

so what we will get is  $\tan^{-1} x + 0 = \tan^{-1} x$  is less than  $\tan^{-1} x + \tan^{-1} \frac{1}{x}$  and that is less than  $\tan^{-1} x + \tan^{-1} \frac{1}{x}$  and of course  $\tan^{-1} x$  here is greater than zero

so what we saw from the previous slide is that for all  $x > 0$   $\tan^{-1} x + \tan^{-1} \frac{1}{x}$  is actually equal to  $\frac{\pi}{2}$  and therefore we clearly see that if  $x$  and  $y$  satisfy this condition then  $\tan^{-1} x + \tan^{-1} y$  which is here is positive and it is less than  $\frac{\pi}{2}$ .

so essentially for this when  $x$  and  $y$  satisfy this condition then the value of  $\tan^{-1} x + \tan^{-1} y$  belongs to the range set of  $\tan^{-1}$  and therefore because they belong to the range set of  $\tan^{-1}$  what will happen is that  $\tan^{-1} x + \tan^{-1} y$  will be equal to this of course the other condition is that if  $x < 0$  and both  $x$  and  $y$  are negative and this can also be the case also the result is very similar that  $\tan^{-1} x$  and  $\tan^{-1} y$  since  $x$  and  $y$  are negative will both belong to the interval  $(-\frac{\pi}{2}, 0)$  and in this case from here what will follow is  $y > \frac{1}{x}$  because both  $x$  and  $y$  are negative therefore from here we get this condition and then from this condition because

so now we are basically on the negative side of the horizontal axis

so we get  $\tan^{-1} y$  is greater than  $\tan^{-1} \frac{1}{x}$  because the  $\tan^{-1}$  function is monotonically increasing with  $x$  as can be seen from this graph here and because  $x$  is  $ah$  and then from this if we add  $\tan^{-1} x$  to both the sides this of course is less than zero and  $\tan^{-1} x$  is also less than zero

so if we add  $\tan^{-1} x$   $ah$  to this particular inequality what we get is  $\tan^{-1} x$  is greater than  $\tan^{-1} x$  plus  $\tan^{-1} y$  is greater than  $\tan^{-1} x$  plus  $\tan^{-1} \frac{1}{x}$  and this  $\tan^{-1} x$  is obviously negative

so from here we also see that now on the the previous slide we saw that for all  $x$  negative  $\tan^{-1} x$  plus  $\tan^{-1} \frac{1}{x}$  is equal to  $-\pi/2$  and therefore it follows that if  $x$  and  $y$  satisfy these three conditions then  $\tan^{-1} x$  plus  $\tan^{-1} y$  lies between zero and  $-\pi/2$  and this interval zero and sorry if  $x$  and  $y$  satisfy this condition then  $\tan^{-1} x$  plus  $\tan^{-1} y$  belongs to the interval  $-\pi/2$  to  $0$  which is a subset of the range set of  $\tan^{-1}$

so even for this case

so this case is also a subset of this case

so essentially both

so this is case three

so both case one and case three belong to this particular case because if you remember case 1 was also  $x, y < 1$  but both  $x$  and  $y$  positive and case 3 which we just discussed just now is also  $x, y < 1$  but here both  $x$  and  $y$  are negative

so both these cases belong to this condition here and in both cases we saw that the the we showed that  $\tan^{-1} x$  plus  $\tan^{-1} y$  belongs to the interval  $-\pi/2$  to  $+\pi/2$  which happens to be the range set of  $\tan^{-1}$  and therefore this will be equal to this

so one more case which remains if we go back to our table that we saw just now

so we have actually divided this  $x, y < 1$  condition into three basically we first divided into two different conditions

so one condition is that  $x, y$  is less than one but it is greater than zero

so so this case one and case three which we just saw now are basically for this  $ah$  sub condition and then there is another condition which is  $x, y$  is less than equal to zero

so if we take the union of this and this condition we finally get this condition now for this condition we have already shown case 1 and case 3 before now we are going to show that  $\tan^{-1} x$  plus  $\tan^{-1} y$  is equal to this also when  $x, y$  is non positive

so if it is less than equal to zero

so that will prove that for this condition  $x, y < 1$   $\tan^{-1} x$  plus  $\tan^{-1} y$  is equal to  $\tan^{-1} \frac{x+y}{1-xy}$

so we have named it as case five

so i will write

so this i will write as case five here

so when and this is quite easy because when  $x$  and  $y$  is non positive

so one case is that  $x$  is less than equal to zero  $y$  greater than zero  $y$  greater than equal to zero the other is the opposite  $x$  is greater than equal to zero but  $y$  is less than equal to zero

so for case five a because  $x$  is less than equal to zero we see that  $\tan^{-1} x$  will belong to the interval  $-\pi/2$  to zero and  $\tan^{-1} y$  will belong to zero to  $\pi/2$  and then of course since we add  $\tan^{-1} x$  with  $\tan^{-1} y$  from this and this it follows that  $\tan^{-1} x$  plus  $\tan^{-1} y$

$y$  will belong to the interval  $-\pi/2$  to  $\pi/2$

so it is very easy to see that for this case  $\tan^{-1} x + \tan^{-1} y$  will belong to this is the range set of  $\tan^{-1}$  and again ah when  $x$  is greater than equal to zero and  $y$  is less than equal to zero  $\tan^{-1} x$  will belong to zero to  $\pi/2$   $\tan^{-1} y$  will belong to  $-\pi/2$  zero

so even in this case you you will see that  $\tan^{-1} x + \tan^{-1} y$  will belong to the range set of  $\tan^{-1}$  and that is because if you see you know  $0$  to  $\pi/2$  and  $-\pi/2$  to  $0$  are if you take you when we when we add these two we essentially have to take the union of this set with this set and the union is precisely the range set of  $\tan^{-1}$  and the same thing is happening in this case also

so essentially we have shown the first row here

so this was case five

so the union of this and this is ah this condition  $xy < 1$

so what we have shown is that if the product of  $x$  and  $y$  is less than one then  $\tan^{-1} x + \tan^{-1} y$  belongs to the range set of  $\tan^{-1}$  and therefore it is equal to this expression here the other two cases are also not very difficult

so one case is where  $xy > 1$  and both  $x$  and  $y$  are positive so in this case  $\tan^{-1} x$  will belong to zero to  $\pi/2$   $\tan^{-1} y$  will belong to zero to  $\pi/2$  again because both of them are positive and because  $xy > 1$  we have  $y > 1/x$

so since  $\tan^{-1}$  is a monotonically increasing function from here we get this statement that  $\tan^{-1} y$  is greater than  $\tan^{-1} (1/x)$  and then as we did before we will add  $\tan^{-1} x$  on both the sides and this is what we end up getting

so on the right hand side we again get  $\tan^{-1} x + \tan^{-1} (1/x)$  which we had shown that when  $x$  is positive the value of this is equal to  $\pi/2$  and therefore what we see is that ah for this scenario here  $\tan^{-1} x + \tan^{-1} y$  is actually greater than  $\pi/2$  it is greater than  $\pi/2$  and therefore ah for this case what we have is

so let us see if i say  $\theta = \tan^{-1} x + \tan^{-1} y$

so this is for the case number two

so what we have is this is greater than  $\pi/2$  but of course this  $\theta$  has to be less than  $\pi$  because because  $\tan^{-1} x$  and  $\tan^{-1} y$  both have belong to  $0$  to  $\pi/2$

so the largest positive value of  $\tan^{-1} x + \tan^{-1} y$  can only be  $\pi/2 + \pi/2$  which is  $\pi$

so obviously this ah  $\theta$  has to be less than  $\pi$

so ultimately for this case two what we have is that  $\theta$  belongs to the interval  $\pi/2$  to  $\pi$  the problem now is that of course ah  $\tan$  of this  $\theta$  is equal to  $x + y$  over  $1 - xy$

so this follows from our very first slide on the  $\tan^{-1}$  functions in this lecture

so the basic problem is again that we are given that

so this is we are given that  $\theta$  must belong to the interval  $\pi/2$  to  $\pi$  and at the same time it should satisfy this condition this equation that  $\tan \theta = x + y$  over  $1 - xy$

so if you remember this again is a problem of the type that we were discussing in the first few slides that you have  $f$  of  $\theta$

so here  $f$  is the  $\tan$  function equal to some  $z$

so  $z$  is basically this value and we need but the problem is that this  $\theta$  does not belong

so  $\theta$  does not belong to the range set of  $f^{-1}$

so even here we say we see that  $\theta$  actually belongs to  $[\frac{\pi}{2}, \pi]$  which is not the range set of  $f^{-1}$  the range set of  $\tan^{-1}$  is actually  $(-\frac{\pi}{2}, \frac{\pi}{2})$

so this is exactly the type of the pattern of problem that we were discussing in the first few slides and you will keep encountering or you will keep seeing this type of problem when you whenever you deal with inverse trigonometric functions

so in this case the trick that we do is first of all we see that since this is not the range set of the  $\tan^{-1}$  function we need to do something

so that some operation on  $\theta$

so so that we basically have

so so what we do is we take this  $\theta$  and we actually subtract  $\pi$  from it

so if we subtract  $\pi$  from it then because  $\theta$  belongs to  $[\frac{\pi}{2}, \pi]$  what will happen is that  $\theta - \pi$  will belong to the interval  $(-\frac{\pi}{2}, 0]$  and what we know is that  $(-\frac{\pi}{2}, 0]$  obviously belongs to

so this interval is a subset of the range set of  $\tan^{-1}$  and the reason why

so the reason for subtracting  $\pi$  from this  $\theta$  is twofold the first reason is that i get a new angle  $\theta - \pi$  which belongs to the

so this belongs to

so this interval is a subset of the range set of  $\tan^{-1}$

so so the reason for subtracting  $\pi$  is two fold again the first reason is that i get a new angle  $\theta - \pi$  which belongs to the range set of  $\tan^{-1}$  the second is that i know that the  $\tan$  function is periodic with  $\pi$

so  $\tan(\theta - \pi)$  is the same as  $\tan \theta$  which we know is equal to  $\frac{x}{1 - y}$

so that is the other reason why we chose this angle  $\pi$  to be subtracted from  $\theta$

so at the end what we have in this equation here is that we have  $\tan$  of an angle where this angle belongs to the range set of  $\tan^{-1}$  equal to this value  $\frac{x}{1 - y}$  and therefore now we can easily write that  $\theta - \pi$  is equal to  $\tan^{-1}(\frac{x}{1 - y})$  and then from here it follows that  $\theta$  equal to  $\pi + \tan^{-1}(\frac{x}{1 - y})$  and that is what was exactly written in the table here that

so this was case number two

so if  $x$  and  $y$  are both positive and  $x > y$  is greater than one then  $\tan^{-1}(\frac{x}{1 - y}) + \tan^{-1} y$  will be equal to  $\pi + \tan^{-1}(\frac{x}{1 - y})$

so that is what we had just proved now and in the interest of time ah we will not go ahead with the proof of this last statement but that can also be done exactly in a similar way that we did case two and then of course from  $\tan^{-1}(\frac{x}{1 - y})$  it is very easy to derive expressions for for example  $2 \tan^{-1}(\frac{x}{1 - y})$

because we can write  $2 \tan^{-1}(\frac{x}{1 - y})$  as  $\tan^{-1}(\frac{x}{1 - y}) + \tan^{-1}(\frac{x}{1 - y})$  and then we can use the  $\tan^{-1} x + \tan^{-1} y$  formula with  $y$  equal to  $\frac{x}{1 - y}$  and this is what we are going to get in that case

so you can its basically going to be a little exercise for you to verify this again from the  $\tan^{-1} x + \tan^{-1} y$  it is easy to get the expression for  $\tan^{-1} x - \tan^{-1} y$  and i will quickly go through it

so  $\tan^{-1} x - \tan^{-1} y$  can be written as  $\tan^{-1} x + \tan^{-1}(-y)$  and that is because the  $\tan^{-1}$  function is an odd function and then we

so now that we have written it ah this thing as a sum of the  $\tan^{-1}$  of two different variables  $x$  and  $-y$

so we should be able to use our result that we had earlier if we use that table then what we see is that we had these three conditions one was

so the first condition is  $x$  times minus  $y$  less than one

so let me show you that table very quickly

so its very simple what we have to just do is that in this whole table everywhere we have to just replace this  $y$  by minus  $y$  and then we will get the expression for  $\tan^{-1} x$  minus  $\tan^{-1} y$

so that is what we have done here

so if you replace  $y$  with minus  $y$  in that table this is what you get

so  $x$  times minus  $y$

so earlier we had  $x y$  less than one but we replace  $y$  with minus  $y$

so we get  $x$  times minus  $y$  less than one which can be written  $x$  the same condition can be equivalently written as  $x y$  is greater than minus one and then then we had earlier we had on the previous slide  $\tan^{-1} x$  plus  $y$  but then instead of  $y$  we write minus  $y$

so we get  $x$  plus minus  $y$  over one minus  $x y$  is what we had earlier but because  $y$  gets replaced with minus five what we have now is  $1$  minus  $x$  times minus  $y$  and the same thing we have to keep on doing for the other conditions and

so finally we end up getting

so these three are the rows for the conditions and the values for  $\tan^{-1} x$  minus  $\tan^{-1} y$  depending on the condition satisfied by both  $x$  and  $y$

so so far we have only seen how to add the  $\tan^{-1}$  of two different variables  $x$  and  $y$  but of course these things can be generalized to for example let us say that we want to compute  $\sin^{-1} x$  plus  $\cos^{-1} y$  and how do we do it

so one one method could be that we can we can obviously find out conversion formulas from  $\sin^{-1} x$  to  $\tan^{-1}$  of something and again from  $\cos^{-1} y$  to  $\tan^{-1}$  of something

so essentially we convert both these terms here to in  $\tan^{-1}$  of two other variables and then since we already know from the previous slide as to how to add the  $\tan^{-1}$  of two variables we will get ah we can get this whole thing as a  $\tan^{-1}$  of one variable and then that should help us

so the problem is that for doing that we would need some conversion formulas from  $\tan$  from  $\sin^{-1}$  to  $\tan^{-1}$  and vice versa and then from  $\cos^{-1}$  to  $\tan^{-1}$  and then from  $\tan^{-1}$  to  $\cos^{-1}$

so in the following slides we will try to quickly ah go over them

so let us take the case of  $\sin^{-1}$  and  $\tan^{-1}$

so so the the question here is that we are having a value of  $x$  ah which obviously is uh between minus one and plus one and then we want to see ah  $\sin^{-1} x$  is equal to  $\tan^{-1}$  of what

so we want to find this question mark here

so let us say that  $\theta$  is equal to  $\sin^{-1} x$  then obviously  $\theta$  belongs to the range set of  $\sin^{-1}$  which is minus  $\pi/2$  to plus  $\pi/2$  and then since we want to find this unknown thing here we are interested actually in finding  $\tan$  of  $\sin^{-1} x$  because if you if you see this equation here if i take the tangent function on both the sides what i will end up getting is  $\tan$  of  $\sin^{-1} x$  if i take the  $\tan$  on both the sides is equal to ah this unknown thing here

so and since  $\sin^{-1} x$  is denoted by  $\theta$  we eventually want to find  $\tan$  of  $\theta$  but  $\tan$  of  $\theta$  is  $\sin \theta$  over  $\cos \theta$  from here we already see that  $\sin \theta$  is actually equal to  $x$  and  $\cos \theta$  is easy to find out because in this case because  $\theta$  belongs to minus  $\pi/2$  to plus  $\pi/2$   $\cos \theta$  will always be positive it will always be positive and therefore and also that we already know this formula that  $\cos^2 \theta$  is equal to one minus

$\sin^2 \theta$   $\sin \theta$  is  $x$

so we can write  $\cos^2 \theta$  as  $1 - x^2$  and then of course  $\cos \theta$  is equal to the positive square root of  $1 - x^2$  because we know that when  $\theta$  belongs to this interval  $\cos \theta$  is greater than zero

so we take the positive square root of  $1 - x^2$  and that is what we put in here

so therefore we finally get that  $\tan \theta$  is equal to  $x$  over root of  $1 - x^2$  and

so if you want to write it cleanly we can write it as  $\tan^{-1} \left( \frac{x}{\sqrt{1-x^2}} \right)$  equals  $\sin^{-1} x$  and because now this if you see this sign inverse  $x$  sine inverse  $x$  will obviously belong to this interval and this interval also happens to be the range set of  $\tan^{-1}$

so we have  $\tan \theta$  equals this value and this  $\theta$  belongs to the range set of  $\tan^{-1}$  and therefore it is very easy to see that obviously this  $\theta$  will be equal to  $\tan^{-1} \left( \frac{x}{\sqrt{1-x^2}} \right)$

so this question mark over here is precisely equal to  $\frac{x}{\sqrt{1-x^2}}$  and then we will do we have to do the reverse also which is that ah suppose if we are given some value of  $x$  ah belonging to real numbers and then we are asked this question that find this argument of the sign inverse function such that  $\tan^{-1} x$  is equal to sine inverse of this unknown variable here

so again we start in a similar manner as what we did earlier let us denote  $\tan^{-1} x$  by  $\theta$  then obviously  $\theta$  belongs to the range set of  $\tan^{-1}$  which is the open interval from  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$  and then  $\sin \theta$  can be written as  $\frac{\tan \theta}{\sec \theta}$  because  $\sec \theta$  is  $1 + \tan^2 \theta$  which is equal to  $\frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$

so we know this identity that  $1 + \tan^2 \theta = \sec^2 \theta$

so this is what we have used for this step over here

so then we can write since  $\theta$  is defined like this it is obvious that  $x$  is actually equal to  $\frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$

so we replace this  $\tan \theta$  by  $x$  here and then  $\tan^2 \theta$  will get replaced by  $x^2$  obviously ah this square root here is the positive square root the reason being that when  $\theta$  belongs to the interval  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$   $\cos \theta$  and therefore  $\sec \theta$  is positive for all  $\theta$  in this interval and therefore since ah  $\sec \theta$  is positive we will take the positive square root of  $1 + x^2$

so finally what we have is that sine of  $\theta$  where  $\theta$  is  $\tan^{-1} x$  is equal to this

so we can actually write it as  $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$  but this angle  $\tan^{-1} x$  belongs to  $-\frac{\pi}{2}$  to  $\frac{\pi}{2}$  which is also the range set of sine inverse it is

so this is actually a subset of the range set of sine inverse and therefore from this statement here it follows that  $\tan^{-1} x$  is equal to sine inverse of  $\frac{x}{\sqrt{1+x^2}}$

so this big question mark here inside the argument of the sign inverse function is actually equal to  $\frac{x}{\sqrt{1+x^2}}$  a little example would help us to realize the importance of this formula that we just derived now

so let us say that we are asked to compute the value of sine inverse of  $\frac{1}{3}$  plus sine inverse of this other real number which is  $\frac{2}{3}$  times one minus square root of  $\frac{1}{3}$  over square root of eight

so if we remember we just proved this formula that sine inverse of  $x$  equals  $\tan^{-1} \left( \frac{x}{\sqrt{1-x^2}} \right)$

so with  $x$  equal to  $\frac{1}{3}$  and if you use this formula then what we will end up getting is sine inverse of  $\frac{1}{3}$  is equal to  $\tan^{-1} \left( \frac{1/3}{\sqrt{1-(1/3)^2}} \right)$

one over three divided by square root of one minus one by nine because one over three whole square is one by nine and then this will be equal to tan inverse of one by square root of eight

so sine inverse of one over three is actually tan inverse of one over square root of eight and this is the positive square root if we go back we you recollect that we had taken the positive square root here and similarly for the other term also we can write that sine inverse of two by three times one minus one over square root of eight equals tan inverse of

so this is x

so  $2 \text{ by } 3 \text{ times } 1 \text{ minus } 1 \text{ over square root of } 8 \text{ divided by the positive square root of } 1 \text{ minus the square of this thing which will be } 4 \text{ by } 9 \text{ times } 1 \text{ minus } 1 \text{ by square root of } 8 \text{ whole square and that can be simplified to tan inverse } 2 \text{ into } 1 \text{ minus root } 8$

so we will take this 3 down into the square root

so that becomes  $9 \text{ minus } 4 \text{ times the square of this particular term here which will be one plus one by eight minus two by square root of eight and that if we take it to the next slide will be equal to tan inverse of } 2 \text{ times } 1 \text{ minus } 1 \text{ by square root of } 8 \text{ over square root of}$

so 5 and then minus half and then plus 8 by square root of 8 which is equal to tan inverse of 2 times 1 minus

so we can actually take the square root of a down and see what we get

so well this can be also written as  $2 \text{ times one minus one over square root of eight over square root of nine by two plus square root of eight and that is equal to tan inverse of two into square root of eight minus one by square root of}$

so we take the square root of eight to the denominator here we get thirty six plus eight times square root of eight

so that will be eight times square root of eight

so this is

so this is for the other term

so this particular term here sine inverse of two by three times this is equal to tan inverse of this thing and then

so now we have to

so we earlier saw that sine inverse of one over three was calculated to be tan inverse of one over root eight and we got another expression for this thing which is here

so now we should be in a position to add both of them

so finally what we have is that sine inverse of 1 over 3 plus sine inverse of 2 by 3 times 1 minus 1 over square root of 8 is equal to tan inverse 1 over square root of 8 plus tan inverse

so in this expression we can cancel this 4 with we can take square root of 4 here

so we get square root of 8 minus 1 by square root of 9 plus two times square root of eight

so now what we have done essentially is we have converted the sum of two sine inverses into the sum of two tan inverses and then we use the tan inverse x plus tan inverse y formula

so in this case what we you can check and that i will not do now is that

so we have x equal to this value and y equal to this value and you can check that the product of this x and this y is less than 1

so that means that i can use for tan inverse x plus tan inverse y i can use this formula tan inverse x plus y by 1 minus x xy this is because this times this which is x times y is less than 1 and then it becomes very easy because then we can write it as tan inverse of 1 by square root of 8 plus root eight

minus one by root of nine plus two root eight over one minus and then one by square root of eight

so it becomes minus root of seventy two plus two times

so plus sixteen times square root of eight and this ah upon further simplification ah whatever is inside here will be one actually

so the numerator and denominator will be the same and that is ah left as a easy exercise

so this will become tan inverse of one which is equal to pi by four

so in this lecture we essentially derived some more relations between these inverse trigonometric functions and specifically we discussed the the formula for tan inverse x plus tan inverse y and the the conversion between sine inverse and tan inverse

so in the next class we will see some other ah formulas and take up some problems also another very important thing which we also saw in this lecture is that we showed how to deal with the cases where we have  $f(\theta) = x$  where  $f$  is some trigonometric function but  $\theta$  is constrained to be not in the range set of  $f^{-1}$  thank you