welcome to the second lecture on inverse trigonometric functions in the first lecture we had defined the basic in the inverses of the basic trigonometry functions like sine inverse x cos inverse x tan inverse x for tangent inverse secant inverse and cosecant inverse in this lecture we are going to study some identities and relations between these inverse functions as a quick recap here is a table of the domain and the range of all the six inverse functions that we had discussed in the previous lecture and we will be referring to this slide every now and then in this lecture also before we start ah here is a little word of caution for most students who tend to confuse between sine inverse x and sine x inverse please make note that these two are not the same for example let us take x equal to zero we know that sine inverse of zero is equal to zero but let us compute the right hand side with x equal to zero sine x inverse is sine zero inverse which is one upon sine zero which is one upon zero which is not defined and therefore this is a good example to illustrate that these two are not the same another thing is an example here is to find the principal value of the tan inverse of minus square root of three now we know that the tan inverse function is has a domain of ah all the real numbers and the range is the open interval between minus pi by 2 and pi by 2 for this particular example here with ah x equal to minus root over three since we know that tan of minus sixty degrees which is minus pi by three

so tan of minus pi by three is equal to minus root three and minus pi by three lies in this interval minus pi by two to plus pi by two therefore we can write that tan inverse of minus square root of three is equal to minus pi over three one ah thing that we have to be careful is that even tan of pi minus pi by 3 is also equal to minus root 3

so tan of this is 2 pi by 3 is also minus square root of 3 but then tan inverse of minus root three will not be equal to this because two pi by three does not belong to the ah to the range set of the tan inverse function

so we have to be a little careful about this thing

so here is the ah first type of identity that we are going to study

so let us say that we have this variable x which lies in the closed interval minus one to plus one and we would like to see what is sine of sine inverse x now let us say that for any x belonging to the closed interval minus one to plus one lets say that sine inverse x is equal to theta now from the range and domain ah as we defined in the last lecture for the sine inverse function we are sure that sine inverse x which is theta

so this particular theta will belong to the closed interval minus pi by two two plus pi by two because the range of this sinus function is minus pi by two two plus pi by two and from here we know that now lets apply the ah sign on both the sides

so sine of

so here

so this is an angle sine inverse x is we denote it by this theta therefore sine of sine inverse x will be equal to sine of theta but when we say that when we say that over here when we say that sine inverse x is equal to theta what does it mean

so as we had defined sine inverse x in the first lecture we said that sine inverse for any x in the interval minus one to plus one is the unique value or the unique ah angle in the range minus pi by two to plus pi by two such that sine of that angle which is theta should be equal to x

so this is how we had defined the sine inverse function and therefore the moment we say that sine inverse x is equal to theta from here it automatically follows that sin theta is equal to x

so this follows from the the way we had defined the sign inverse function in

the first lecture and therefore when we combine this statement ah with this statement what we get is that this is equal to x and therefore sine of sine inverse x is equal to x

so here we have our first result that sign of sin inverse x for any x in the closed interval minus one to plus one is equal to x but what about the other way around

so this leads us to the other question which is for all y is it true that sine inverse of sine y equals y and we can immediately check this for example let us say we take y equal to pi over 6 for which sine of pi over 6 which is 30 degrees is equal to half and from the graph of the sine inverse function we know that sine inverse of half is equal to pi over 6

so for y equal to pi over six this statement is indeed true but what if we take let us say y equal to two pi over three then sine of two pi over three equals
so two pi over three is uh one twenty degrees

so sine of two pi over three will be root three over two and sine inverse of sine y with y equal to two pi over three will be sine inverse of root three over two which is actually equal to pi over three

so naturally ah in this example for example in this particular example we see that y was 2 pi by 3 but sine inverse of sine y is only pi by 3.

it is clear that ah from the range of the sine inverse function we know that the range of the sign inverse function is minus pi by 2 to plus pi by 2 and therefore from therefore sine inverse of sine y for any y sine inverse of sine y has to belong to minus pi by two two plus pi by two which is the range set of the sine inverse function and therefore as we will show now if y belongs to if y belongs to this range set then it is true that sine inverse of sine y equals y however it is clear that if y does not belong to this range set then y cannot be equal to sine inverse sin y because sine inverse sine y for any y will have to belong to minus pi by 2 to plus pi by 2 and if y does not belong to this set it is not possible that y and sin inverse sin y be equal we now show that for any y belong to the range set of sine inverse which is minus pi by two to plus pi by two

so for any y belonging to this interval sine inverse of sine y is exactly equal to y

so lets start with sine inverse of sine y and let it be equal to x then clearly x belongs to the range set of the sign inverse function which is minus pi by 2 to plus pi by 2 and then we take the sign on both the sides of this particular equation

so what we then get is that sine of sine inverse sine y equals sine x and from the previous slide we already know that for any z sine of sine inverse of z is equal to z for all z such that mod of z is less than equal to one

so we are going to use this particular result ah in this equation with z being equal to sine y

so we will treat this as z and therefore sine of sine inverse z is equal to z and hence this left hand side is equal to sine y which is z

so starting from here we finally have sine y equal to sine x since both x and y belong to the interval minus pi by 2 to plus pi by 2 and also in this interval if we draw it

so here we draw the sign x versus x only within the interval x belonging to minus pi by two to plus pi by two lets say this is minus one and this is plus one then we know that the the graph of the sin x functions function the graph of the sine x function is going to look something like this and then we have two values x and y

so both x also belongs to the interval minus pi by two to plus pi by two y also belongs to the same interval here

so we are given that both x and y belong to this interval and we are said that it is said that sine x equals sine y but then we clearly see that within this interval the graph of the sine function is monotonically increasing its monotonically increasing and therefore if sine x is equal to sine y then it must be true that x is equal to y and therefore we have x to be equal to y which proves that as long as y belongs to the range set of sine inverse sine inverse of sine y is equal to y

so what we just said now was that if we are said that y belongs to minus pi by two to plus pi by two then sine inverse of sine of y will always be equal to y but this statement is not true if y is outside this closed interval a similar thing actually holds for ah all the other inverse functions and we will not prove them in the interest of time

so for example i have just written it here in this slide

so we have for any value of x in the interval in the closed interval minus one to plus one cos of cos inverse x is equal to x

so so this particular statement is very straight forward but when you take the ah other case where you compute cos inverse

so here you are composing cos with cos inverse and here you are doing the reverse you are composing cos inverse with cos

so if you take cos inverse of cos theta then it is equal to theta only for those theta which belong to the range of the cos inverse function the range of the cos inverse function as we defined it in the first lecture was the closed interval zero to pi

so as long as the theta belongs to this closed interval this statement is correct but the moment we take any theta which lies outside this interval then we cannot say that this is true and a similar thing holds for the tangent function the cotangent function and also for the the secant function and the cosecant function also

so as you can see that secant inverse of sec theta is theta only for those theta that belong to the

so only for those theta such that they belong to the range set of the secant inverse function and similarly here this statement here is true only if theta is such that it belongs to the range of the cosecant inverse

so this is the range of the cosecant inverse function let us discuss the second identity now

so we will now show some relation between the sine inverse function and the cosecant inverse function we already know that and the motivation for this is that we already know that cosecant of x for any x real is one by sin x

so that makes us believe that there should be some relation between cosec and inverse and sin inverse also and this is the relation now let us take any x such that

so here x is such that mod of x is greater than equal to one because we are going to consider cosecant inverse of x and we know that the domain of the cosecant inverse function is all x

so that their absolute value is greater than equal to one that is why we have considered only this range of values for x now let us say that cosecant inverse of x is equal to some angle theta now from the definition of the range of the cosecant inverse function as we ah discussed in the previous lecture we know that this theta must belong to the closed interval minus pi by two to plus pi by two except the value zero because

so this is actually the range set of the cosecant inverse function and therefore theta has to belong to this but from this statement what we can write is that if we if we apply the cosec ah function on both the left and the right hand side

so we are essentially trying to use this particular identity now

so so we apply cosec on both the sides then we have cosec of cosec inverse x is equal to cosecant theta and then from this identity we have cosecant of cosecant inverse x equals x as long as mod of x is greater than equal to one here we already have mod of x greater than equal to one and therefore this left hand side here should be equal to x and therefore we have x equal to cosecant of theta which is actually equal to from the definition of the cosecant function this is equal to one by sin theta or that can also be written as sine theta equals one by x now we also from here since mod of x is greater than equal to one from this statement it follows that one by x has to be less than equal to one and we have sin theta equal to one by x further we also know that theta belongs to this set and therefore theta for sure belongs to the

so this particular ah range set is a subset of the range set of the sine inverse function

so we know that from from this statement we can also say that theta belongs to minus pi by two two plus pi by two now minus pi by two to plus pi by two is the is the range set of the sine inverse function and therefore we can say that since sin theta equal to one by x and theta belongs to minus pi by two to plus pi by two

so from here we can say from the one of the previous identities that we had discussed just a few slides back we have shown that as long as y belongs to this interval closed interval sine inverse of sin y is y

so instead of y lets take theta

so because theta already belongs to this ah minus pi by two to plus pi by two interval from our previous result we can say that sine inverse of sine theta is equal to theta

so this is from the result we just discussed now what we do is we apply the sine inverse function on both the left hand side and the right hand side of this equality

so by applying sine inverse function to this equality what we get is sine inverse of sine theta is equal to sine inverse of one upon x but from here what do we have we have that this is exactly equal to theta and therefore we finally have that theta is equal to sine inverse one upon x and from the previous slide we had said that theta is actually cosec inverse of x

so this is also equal to cosec inverse of x and therefore we have shown that for any x such that mod of x is less than equal to sorry mod of x is for any x such that mod of x is greater than equal to one we have finally shown that cosecant inverse of x is equal to sine inverse of one over x in a similar manner we can also show that cos inverse of one over x is equal to secant inverse of x for all x greater than equal to one and the proof is over here

so the steps are similar

so we just have to take let us say that the lhs is equal to theta but since the range of the cos inverse function is from zero to pi it obviously follows that this theta must lie in the interval zero to pi and then we apply the cos function on both the left and the right hand side here

so which we get cos of cos inverse one by x is equal to cos of theta but cos of cos inverse one by x is simply one by x from the result we had from one of the previous slides

so we get one by x equals cos theta which is the same as saying that sec theta is equal to x because sec theta is one over cos theta

so from there it follows that sec theta is x now since theta belongs to zero to pi and the range set of secant inverse

so we know that the range set

so this range set of the secant inverse function is 0 to pi minus pi by 2

so all the points in the close interval zero to pi except pi by two now this theta is already there in this interval zero to pi

so therefore this theta is definitely going to belong to the range set of the secant inverse function and therefore it automatically follows that since x sec theta is equal to x sec inverse of x will be equal to theta but theta was already equal to cos inverse of one over x and therefore by combining this and this we get that sec inverse x and cos inverse one upon x are the same and there is a similar ah relation between tangent inverse of 1 upon x and cotangent inverse the proof is again along the similar lines but there is a slight twist here we say that this is this thing is applicable only for positive values of x and that will become clear as we go in down the proof now since x is greater than 0 1 upon x is also greater than 0 and therefore tan inverse let us say if we say that tan inverse 1 upon x is theta the range of the tan inverse function is the open interval between minus pi by two to plus pi by two since one upon x is greater than zero tan inverse of one upon x will also be positive and therefore surely this theta has to belong to zero to pi by two because the range of the tan inverse function is only between minus pi by two to plus pi by two and one upon x is greater than zero

so then we have this statement here but if we applied the tan function to both the left and right hand side what we get is tan of theta is one upon x now tan theta is one upon cot theta

so from that relation we get that cot of theta is x and we have x greater than zero the range set of the cot inverse function is zero to pi and that we know from our last lecture from here from this statement we know that we had shown that this theta will belong to 0 to pi by 2 but 0 to pi by two is already a subset of the range set of cot inverse which is zero to pi and therefore it follows that this theta here which is tan inverse of one upon x has to belong to the interval zero to pi

so now what we have is we have cot of theta equal to x where this theta is belonging to the range of the cot inverse function and therefore it automatically follows that cot inverse of x will be exactly equal to theta this is only happening because theta belongs to the range set of the cot inverse function which is zero to pi if theta were to be outside this range set of the chord inverse function then this identity would not have been true

so then finally we have theta to be equal to both cot inverse x and tan inverse one upon x and hence both of them are equal but see that this is only 2 when x is greater than 0 the problem with x negative or equal to 0 is that

so if you take x negative then what will happen is that 1 upon x will be negative which will imply that theta will belong to the interval in that case where the x is negative theta will not belong to this interval instead it will belong to the interval minus pi by 2 to 0 because x is negative therefore one by x will also be negative

so in that case theta will belong to minus pi by two to zero which is and this range or this interval is not a does not belong to or is not a subset of the range set of cot inverse therefore even in that case this and this statement will be true even when x is negative these two statements will be true but what will be not true is that

so we will have a problem at this step over here because minus pi by two the range of this ah rather the interval to which this theta belongs which is minus pi by two to zero will not be a subset of will not be a subset of zero to pi it will not be a subset of zero to pi which is the range set of cot inverse and therefore we cannot say that theta equal to cot inverse x

so that is why for negative x this will this equality will not hold

so lets see the relation between sine inverse of minus x and sine inverse x for

obviously the x here because it is actually the domain of the sine inverse function

 so we did not write but

 so x has to belong to the closed interval minus one to plus one now we know that sine x is an odd function and we will quickly quickly see that the same is true for sine inverse of x

 so lets say that sine inverse of minus x equals theta since the range of the sine inverse function is the close interval minus pi by two to plus pi by two it follows that theta must belong to that range set minus pi by two to plus pi by two close interval and then we take the sign on both the sides of this equality we get sine of sine inverse minus x is equal to sine of theta but this left hand side over here is exactly equal to minus x and therefore what we get is minus x is equal to sine of theta which from where we can also write that x is equal to minus sign of theta now we know that the sine function is an odd function

 so we know that for any theta sine of minus theta is equal to minus sine theta

 so this right hand side over here is equal to sine of minus theta

 so finally we have this equation that x is equal to sine of minus theta now since theta belongs to this interval minus theta will also belong to the same interval

 so minus theta is also going to belong to the range set of the sine inverse function

 so this since this minus theta belongs to the range set of the sine inverse function and x is equal to sine of minus theta what we get is that minus theta is equal to sine inverse of x which can be then written also as theta is equal to minus of sine inverse x and therefore by combining this with this what we eventually get is that sine inverse of minus x is equal to minus of sine inverse x

 so even for the sine inverse function we see that it is an odd function a similar thing is ah true for the tan inverse function and we can quickly show that

 so suppose tan inverse of minus x is theta what that means is that the theta belongs to the over in open interval minus pi by two two plus pi by two and therefore tan of theta is equal to minus x ah by applying tan on both the sides here and then ah from here what we can say is that x is equal to minus tan theta which is equal to tan of minus theta because even the tan function is an odd function

 so the tan function is an odd function

 so this is true but since theta belongs to minus pi by two to plus pi by two minus theta will also belong to the same interval and therefore minus theta belongs to the range of the tan inverse function for from where it follows that minus theta is equal to tan inverse of x

 so from this equation it follows that this minus theta is tan inverse of x because this minus theta minus theta belongs to the range set of the tan inverse function and which can also be written as theta equal to minus of tan inverse x and therefore from here and here what we see is that tan inverse of minus x equals minus of tan inverse x and similarly ah the other identities ah can also be ah shown for example for cos inverse minus x and cos inverse x

 so i will go over it a little faster

 so this belongs that theta belongs to zero to pi and therefore cos theta is equal to minus x that is same as writing x equal to minus cos theta but minus we know that for any theta cos of pi minus theta is equal to minus cos theta

 so this is equal to

 so x is equal to cos of pi minus theta now since theta belongs to zero to pi this pi minus theta will also belong to the same interval zero to pi and

therefore from here it follows that pi minus theta is equal to cos inverse of x is equal to cos inverse of x and cos inverse of minus x was theta

so if instead of this theta here if we write it as cos inverse of minus x from this equation what we get is pi minus cos inverse of minus x is equal to cos inverse of x from where we finally get that cos inverse of x plus cos inverse of minus x is equal to pi and this is true for any x such that mod of x is less than equal to one

so the main step this was the main step here where we found that minus cos theta is actually cos of pi minus theta and it also turned out that this because theta belongs to zero to pi pi minus theta would also belong to the same interval zero to pi and this zero to pi is the range set of the cos inverse function

so therefore since x equal to cos of pi minus theta and this angle pi minus theta is already in the range set of the cos inverse function this statement follows and then from there it was very easy ah the relation between cosec inverse minus x and cosec inverse x is the same as that for the sine function and we can go over very quickly it should be not very difficult for you to understand

so let us say this is equal to theta then theta has to belong to the range set of the cosec inverse function which is the closed interval minus pi by two to plus pi by two except the element zero and therefore from there if we take cosec on both the sides left and the right hand side of this above equation we get cosec of theta is minus x which essentially implies that x equal to minus of cosec of theta but cosec is an odd function and therefore minus of cos x theta is the same as cosec of minus theta equals x now if theta belongs to this interval then minus theta will also belong to this interval and therefore since minus theta belongs now this interval is basically the range set of cosec inverse and therefore minus theta belongs to the range set of cosec inverse and hence from this statement it follows that minus theta is equal to cosec inverse of x but theta was already equal to cosec inverse of minus x and therefore from here it follows that theta is equal to minus cosec inverse x but theta is this thing and hence we can conclude that cos x inverse of minus x which is theta is equal to minus of cosec inverse x but remember that this is only because the domain of the cosec inverse function is only those x such that mod x is greater than equal to one and what we see is that just like the sine function the cosec inverse function is also an odd function because it follows from here in the interest of time we are not going to prove the next two statements and it is left to you as a little exercise and what we have here is again the relation between secant inverse of minus x and secant inverse of x and it can be shown that for all mod of x greater than equal to one

so this this particular set here is actually the domain of the secant inverse function

so you take any x in the domain of the secant inverse sec inverse of minus x is equal to pi minus sec inverse of x which is similar to the relation between the cos inverse of x and cos inverse of minus x and a similar string is also true for the cot inverse function we already know that the sine and cosine functions are essentially the same ah they are just shifted versions of each other

so it will be interesting to see if we can derive some relation between sin inverse x and cos inverse x and this obviously since the domain of the sine inverse and the cos inverse function is the same which is that mod of x should be less than equal to one we should be able to define some relation between them

so let us start with saying that let us say that sine inverse x is equal to theta from where it follows that theta must belong to the range of the sine inverse function which is the closed interval minus pi by two two plus pi by two

if we apply sign to this equation on both the sides what we get is sine of sine inverse x equals sin theta but this left hand side is essentially x
 so what we end up getting is x equal to sine of theta but we know that from our initial lectures on trigonometric functions we know that sine of theta is equal to cos cos of pi by two minus theta this relation is already known to us and therefore from using this relation here what we get is that x is equal to cos of pi by two minus theta now since theta belongs to the interval minus pi by two to plus pi by two it follows that pi by two minus theta will belong to the interval to the closed interval 0 to pi and what we have now is that we have x equal to cos of some angle and this angle actually belongs to the set 0 to pi the closed interval 0 to pi but the close interval 0 to pi is actually the range set
 so the closed interval 0 to pi is in fact the range set of the cos inverse function and since this pi by 2 minus theta belongs to the range set of the cos inverse function it follows from here that pi by 2 minus theta equals cos inverse of x and therefore if we combine this statement here and this statement what we see is that pi by 2 minus now theta was sin inverse of x
 so pi by 2 minus sine inverse x equals cos inverse x from where we can finally conclude that sine inverse x plus cos inverse x equals pi by 2 for all x such that mod of x is less than equal to 1.

 so this is a very fundamental identity which you must remember in the same manner and the style of the proof being the same you can also show that tan for any x real value tan inverse x plus cot inverse x is equal to pi by two and also that for any x such that mod of x is greater than equal to one
 so this this set or this the set of values for x is actually the domain function for both the secant inverse as well as the cosecant inverse function and therefore it you can also show that secant inverse of x plus cosecant inverse of x equals pi by two
 so this is left as an exercise for you
 so now let us see if we can for any given x and y
 so here both x and y are in the domain set of the tan inverse function which is nothing but the set of all real numbers
 so suppose if we say that x and y are both real then can we ah simplify this and write it as tan inverse of something where this something here depends on both x and y
 so let us say that tan inverse x is equal to theta and tan inverse y is equal to phi
 so automatically it follows that both theta and phi must belong to the range of the tan inverse function which is the open interval between minus pi by 2 n plus pi by 2 and then what we want to compute is theta plus phi
 so what we want is that what to see is whether we can write theta plus phi as tan inverse of something here
 so to do that of course the ah the necessary condition is that if this were to be true then the necessary condition is that tan theta plus phi must be equal to this thing inside the inside the bracket here because if if if this is true then by applying the tangent function on both the left hand and the right hand side what we will get is tan of theta plus phi must be equal to whatever is there as the argument of the tan inverse function
 so but that but this i mean going from here to here is correct but this we cannot necessarily say that let us say that if we say that tan theta plus phi is equal to something
 so let us quickly derive what this is equal to
 so we know that tan of theta plus phi again from our previous lectures ah we know that tan of theta plus phi is tan theta plus tan phi over 1 minus tan theta

tan phi which is further equal to now from here when we define theta and phi to be tan inverse of x and tan inverse of y from each of the statements it will follow that since tan inverse x is theta if we take the tan function on both the sides what we get is x equal to tan of theta from here and similarly from this statement here what we get is y is equal to tan of phi

so we are going to use both of these here

so tan t tan ah tan theta is x

so this becomes x plus tan phi is y x plus y over one minus x y

so what we see is that tan of theta plus phi is equal to x plus y over one minus x plus 1 minus x times y but that does not necessarily mean that

so from from here can we say that theta plus phi is equal to tan inverse of x plus y y divided by one minus x y well this is not necessarily true always this will be true if and only if the sum of these two angles theta and phi belongs to the range set of the tan inverse function

so so far what we had shown was that tan of theta plus phi is equal to x plus y over one minus x y but this does not necessarily imply that theta plus phi equals tan inverse of x plus y one minus x y now this is only true only when theta plus phi belongs to the range set of the tan inverse function which is the open interval two two minus pi by two to plus pi by two

so if if theta plus five where to belong to this interval then this is correct

so if theta plus phi belong to belong to minus pi by 2 plus pi by 2 then it is true that tan inverse of x which was theta plus tan inverse of y which was phi equals tan inverse of x plus y over one minus x y but when is this condition two when is this condition true it can be shown that this condition is true now the problem is that both theta

so we know that theta also belongs to the theta also belongs to the interval minus pi by two to plus pi by two and phi also belongs to this interval

so it is possible that when we add them the sum of these two might not necessarily belong again to the same interval it might go outside the interval but we will show that this is true if and only if

so this condition is true if and only if the product of x times y is less than one

so only then it is true

so essentially what we have shown is that if we are given x and y such that x times y is less than one then it follows that tan inverse x plus tan inverse y equals tan inverse of x plus y over one minus x y here is a nice table if we will show that if x times y is less than one then it is true that tan inverse x plus tan inverse y is exactly equal to ah tan inverse x plus y over one minus x y but

so this this is equal to this only if x y is less than one but for the other cases

so the other cases if both x and y are positive and x y is greater than one in that case we will have to add a pi to this expression

so this is what we get on the other hand if both x and y are negative but the product is still greater than one we add a minus pi

so in the next class we are actually going to start with ah probably showing you why this x into y less than one is a necessary and sufficient condition to ensure that tan inverse x plus tan inverse y which is theta plus phi belongs to the range set of the tan inverse function and then we will also continue with some of the other identities and followed by some problem solving session thank you you