

hello and welcome to the fourth lecture on limits

so in the last lecture we discussed about infinite limits and then we looked at some of the properties of limits with respect to sum of functions and then we started with the product of functions

so let me continue with that

so last time we stopped at this theorem at if limit of f of x is infinite a and limit of g of x is l then and suppose l is non-zero then the limit of the product function $f(x) \times g(x)$ this is equal to infinity if l is positive and this is negative infinity if l is negative

so let me prove this for the case of l positive

so assume l is positive then what we have to show is that

so let m bigger than zero be given we need to find δ positive such that $|f(x) - a| < \delta$ implies $f(x) \times g(x) > m$ right

so first of all what we know is that since limit of g of x as x goes to a is equal to l by taking $\epsilon = l$ and l is positive

so by taking ϵ equal to say $l/2$ which is also positive we can find a δ_1 let me write δ_1 positive such that $|g(x) - l| < \delta_1$ implies $g(x) > l/2$ greater than 0 this implies $g(x)$ in absolute value is less than $l/2$ which is equal to $l/2$ which implies $g(x)$ is greater than $l/2$ but we need only this $g(x)$ greater than $l/2$

so similarly because since limit of f of x is equal to infinity there exists δ_2 positive such that $|f(x) - a| < \delta_2$ will imply $f(x)$ is greater than whatever positive number m one you take and then again we will write what this m one should be

so for that if we take

so if δ is equal to minimum of δ_1 and δ_2 then $|f(x) - a| < \delta$ and $|g(x) - l| < \delta$ will imply that $f(x) \times g(x)$ is greater than m one times $l/2$ and what we want is that this should be greater than m

so we choose m one times $l/2$ equal to m which means that m one should be equal to twice m by l

so if I choose m one to be $2m$ by l then m one times $l/2$ is equal to m and this says that $f(x) \times g(x)$ is greater than m whenever $|f(x) - a| < \delta$

so the limit of $f(x) \times g(x)$ is equal to infinity similarly we can prove for l less than zero

so let me skip that part next we will see what happens if l is equal to zero

so this is more important

so question what if l is equal to zero is the limit equal to zero is it infinity is it something else

so let us consider example first let us take $f(x)$ to be equal to $1/x^2$ and $g(x)$ is equal to say x^2 not equal to 0

so in this case here limit of f of x at 0 is infinity limit of g of x is equal to zero also if we multiply $f(x)$ and $g(x)$ this is equal to one for all x not equal to zero

so therefore the limit of $f(x) \times g(x)$ is equal to one in this case

so here we see an example where the limit of $f(x)$ is infinity limit of $g(x)$ is zero and limit of the product is one now if we change this one by x^2 to

so if we change let me write

so by changing $g(x)$ to any constant times x^2 we see that $f(x) \times g(x)$ is then equal to c for x not equal to zero and therefore the limit of $f(x) \times g(x)$ is equal to the constant c

so we see that for the product if one is going to infinity and the other one is going to zero the limit can be made to be any real number that we want again let's see if we can make it to be infinity can we make it next example we take again $f(x)$ to be equal to one by x square and $g(x)$ to be equal to x

so here again limit of $f(x)$ is infinity limit of $g(x)$ is zero but $f(x)$ times $g(x)$ will be equal to one by x for x not equal to zero and then we see that

so if we calculate the left hand limit of $f(x)$ times $g(x)$ this is equal to negative infinity and right hand limit is positive infinity

so we see that in this case the left hand limits and right hand limits are not even equal and they are infinity or minus infinity if i take $f(x)$ equal to one by x square $g(x)$ equal to say x cube then the limit of $f(x)$ times $g(x)$ $f(x)$ is one by x square $g(x)$ is x cube

so the product is x and this limit is 0

so the limit of infinity times 0 it can be 0 it can be infinity it can be minus infinity it can be any real number

so this is again an indeterminate form conclusion infinity times 0 is also and indeterminate form and the limit depends on the problem right

so you should never make the mistake of writing the limit to be equal to zero just because the limit of one function is zero right you have to see whether the other limit is finite or not another thing is if the limit of both $f(x)$ and $g(x)$ are infinite then we can say what the limit of the product is

so if limit of $f(x)$ is infinity and limit of $g(x)$ is also infinity then limit of the product is infinity and the other one is if one of the limit is infinity and the other is negative infinity if limit of $f(x)$ is infinity and limit of $g(x)$ is negative infinity then limit of $f(x)$ times $g(x)$ is negative infinity and you can have both the limits to be negative infinity then again the product is the product has limit to be equal to positive infinity this is because product of two negative numbers gives a positive number right

so these things again you can prove using the regress definition but they should be intuitively clear because if you have $f(x)$ and $g(x)$ are both large positive number then the product has to be again large positive number if one of $f(x)$ is large positive number another is large negative number then the product will give you large negative number and if both are large negative numbers then the product will be large positive number

so that is all which this theorem is telling but the problem is if you have one is a large neg positive number another one is a small positive number in that case we cannot say whether the product will be a large positive number or a small positive number or it will be some other constant

so let's summarize

so we will write the the limit forms

so if we have infinity one limit is infinity another is some constant then this is always equal to infinity we have minus infinity plus any constant then that is equal to minus infinity note that these are just notation to say that if the limit of one function is infinity another is a constant c then this sum is infinity similarly the limit of one function is negative infinity limit of another function is a constant then the limit has to be negative infinity infinity plus infinity this gives infinity minus infinity then minus infinity that will give you minus infinity and if i have infinity times infinity this is equal to infinity infinity times minus infinity is minus infinity minus infinity times minus infinity is equal to infinity

so in these cases the limits are the determinate these are given to be equal to this also some indeterminate forms we have seen that infinity minus infinity this we cannot say what it is similarly zero times infinity this is an indeterminate form also 0 by 0 this we have seen previously that if the

numerator and denominator the limits are both 0 then the limit of the quotient this depends from problem to problem it can be a real number or it can be 0 or it can be infinity infinity by infinity that is again an indeterminate form and later we will see some more indeterminate forms

so let me give one example of where the limit is infinity for trigonometric functions i am sure you have seen trigonometric functions

so one example is what can we say limit of \tan of x as x approaches $\pi/2$.

so what we know is that we know that \tan of x is $\sin x$ by $\cos x$ also \sin of $\pi/2$ is equal to 1 and \cos of $\pi/2$ is equal to zero

so $\tan \pi/2$ is undefined

so \tan of $\pi/2$ is not defined but \tan of x is defined in a small interval containing $\pi/2$ and

so if we see this another thing we know is that also \cos of x is positive if x is from 0 to $\pi/2$ and \cos of x is negative if x is from $\pi/2$ to π right

so maybe i can draw the graph of \cos of x

so it is z at zero it is one because of x the graph looks like this you have zero $\pi/2$ and then this is π

so \cos of x \cos of $\pi/2$ is zero and it is positive when you are in the interval from 0 to $\pi/2$ it is negative when you are from $\pi/2$ to π in fact till $3\pi/2$ it is negative

so so therefore the limit of $1/\cos x$ as x goes to $\pi/2$ from the left the left hand limit of $1/\cos x$ this will be equal to positive infinity and the right hand limit of $1/\cos x$ this is equal to negative infinity and the limit of \sin of x at $\pi/2$ this is equal to one

so by the product rule we know that therefore the left hand limit of \tan of x this is equal to infinity and the right hand limit of \tan of x is equal to negative infinity this you can also see from the graph of \tan of x which is not defined at this $\pi/2$ and from 0 as you go to $\pi/2$ it goes to infinity then again it is 0 at π and this side it goes like this right

so at $\pi/2$ if you approach $\pi/2$ from the left then \tan of x becomes larger and larger positive real number

so $\tan x$ goes to infinity as x goes to $\pi/2$ from the left whereas if you approach $\pi/2$ from the right then \tan of x becomes larger and larger negative number

so the limit of $\tan x$ the right hand limit of $\tan x$ at $\pi/2$ is equal to negative infinity similarly you can write for \cot of x cotangent x secant x and cosecant x ok

so the next thing i will talk about is what do we mean by limit at infinity limits at infinity and minus infinity

so suppose f of x is a function which is defined for x greater than m or some m positive if it is defined for all x bigger than m then we can talk about the limit of f of x at infinity

so we say limit of f of x as x approaches infinity this is equal to l if given any ϵ positive there exists a positive constant there exists a k positive such that $|f(x) - l|$ this is less than ϵ whenever x is bigger than k right

so limit at infinity just means that this is limit at x equal to infinity is a number l if you can make f of x to be arbitrarily close to l whenever x is very large right

so here l is a real number

so similarly we say limit of f of x as x approaches negative infinity is equal to l a real number if given ϵ positive bigger than 0 there exists some negative number n such that $x < n$ implies $|f(x) - l|$ in absolute value is

less than epsilon right

so the picture looks like this suppose we have some l here this is our y equal to l and then if i choose any epsilon and then if i take l minus epsilon to l plus epsilon and we would say that the limit of f of x at infinity is equal to l provided there exist some k here such that when whenever x is bigger than k f of x should be between l minus epsilon to l plus epsilon

so this is k and the function it can be something like this in this case we will say that the limit is equal to l similarly at negative infinity means if you have there this sum n such that before this n the function has to approach this l as right

so given any epsilon you can choose this n such that the value of the function can be anything here between

so this is what it means now let's see by some examples

so one is simply if we take f x equal to x and what is limit of f of x and limit of f of x as x goes to infinity ah let me not write this

so let let me take f of x equal to one by x

so what is the limit of f of x as x goes to infinity claim

so what happens if you take f x equal to 1 by x then if you take x to be a larger and larger positive real number then 1 by x will become smaller and smaller right

so that means that if you take any positive however small positive real number you can make 1 by x to be smaller than that

so this limit of f of x should be equal to zero let us try to prove it rigorously

so to prove this what we have to show is that given any epsilon

so let epsilon bigger than zero be given then we have to find a k such that if x is bigger than k then f of x is less than epsilon

so if we take k to be equal to 1 by epsilon then x bigger than k equal to 1 by epsilon this implies that f x which is one by x has to be less than epsilon that is mod of f x minus zero

so f of x is equal to one by x and minus zero that is equal to one by x and because we are taking x to be bigger than a positive number k

so f x minus zero this is less than epsilon for all x bigger than k hence the limit of f of x x going to infinity is equal to zero similarly limit of one by x as x goes to negative infinity this is again 0 note that if x is a negative real number then 1 by x is always negative but then it becomes a smaller and smaller negative

so it becomes a a negative real number but very close to zero

so this is also true now let us combine these two things

so combining the definitions

so we have seen one is the definition of limit being infinity and another one is the limit at infinity

so combining the definitions of infinite limit and limit at infinity we can define limit of f of x as x going to infinity to be equal to infinity if given any m bigger than zero there exists some k bigger than zero such that f of x is bigger than m for all x bigger than k right

so we say that the limit of f of x at infinity is infinity if for any large value of x the value of f of x is large similarly we can define the limit at infinity to be minus infinity or limit at negative infinity to be infinity similarly one can define limit f x x going to infinity equal to minus infinity and limit x going to negative infinity of f x equal to infinity and limit x going to negative infinity of f x equal to negative infinity

so example if we take f of x equal to x and limit of f of x as x goes to infinity this is equal to infinity limit of f of x as x goes to negative

infinity this is equal to negative infinity right because we have $f(x)$ equal to x so if x is large positive number then $f(x)$ is also a large positive number and if $f(x)$ if x is a large negative number then $f(x)$ is large negative number if we take $g(x)$ equal to x^2 then here the limit of $g(x)$ as x approaches infinity this is equal to infinity and this is also the limit of $g(x)$ as x approaches negative infinity but lets see some limit at infinity that does not exist

so question limit of say sine of x as x approaches infinity equal to what

so recall that sine of x is a periodic function and sine of any $m\pi$ this is equal to zero for all integer m and sine of if we take any two $m\pi + \pi$ by two then this is equal to one again for all integer m

so now if we see here what happens if sine of x to determine the limit of sine of x what we have to see is that what happens to this function as x becomes larger and larger but what we see is that this function you have x equal to 0π and then 2π and then this keeps repeating right

so no matter how large x you take sine of x keeps oscillating between negative one and one this is one this is minus one

so in this case since sine of x keeps oscillating between negative one and one in any interval of length two π we see that the limit of sine of x as x approaching infinity this does not exist

so this is an example where the limit of function at infinity does not exist right

so here no matter what value of ϵ you take you cannot make the function to be between $1 - \epsilon$ to $1 + \epsilon$ by choosing x to be large because you know that the function will take all the values between negative one and one

so no matter what ϵ you choose it will not be close to remain close to 1 for all large values of x

so that is why the limit of sine of x does not exist similarly the limit of \cos of x as x approaches infinity that also does not exist similarly limit \cos of x as x approaching infinity does not exist again some properties

so suppose limit of $f(x)$ as x approaches infinity is equal to l_1 and limit of $g(x)$ as x approaches infinity equal to l_2 where l_1 and l_2 are real numbers then limit of $f(x) + g(x)$ at infinity is equal to $l_1 + l_2$

so this is just this sum rule for the limit at a point a if the limits are finite l_1 and l_2 then the limit of the sum of the function is sum of the limits l_1 and $l_1 + l_2$

so this can be proved like we have shown for the other cases similarly if we have limit of $f(x)$ is l_1 and limit of $g(x)$ is l_2 then limit of the product $f(x) \cdot g(x)$ is equal to $l_1 \cdot l_2$ also for the quotient the denominator should not be zero

so if limit of $f(x)$ at infinity is l_1 and limit of $g(x)$ is equal to l_2 where l_1 is any real number and l_2 is a non zero real number $l_2 \neq 0$ then the limit of $f(x) / g(x)$ is equal to l_1 / l_2 right

so here l_1 can be zero but l_2 has to be non zero for this conclusion now if the limit l_2 is 0 then we get a indeterminate form

so for example let me do an example

so $f(x)$ equal to say $1/x$ and $g(x)$ equal to $1/x^2$

so here limit of x going to infinity of $f(x)$ is 0 limit of $g(x)$ at infinity that is also 0 what about limit of $f(x) / g(x)$ $f(x) / g(x)$ is equal to $1/x$ divided by $1/x^2$

so this is equal to x and therefore limit of $f(x) / g(x)$ as x approaches infinity this is equal to infinity but if i look at $g(x) / f(x)$ and this is equal to $1/x$ for any x naught equal to zero therefore the limit of $g(x) / f(x)$ this is equal to 0 and by taking $f(x)$ equal to say a constant divided by x and $g(x)$

x equal to one by x we see that we have limit of f of x as x approaches infinity is equal to θ which is also the limit of g of x as x approaches infinity and the limit of f of x by g of x you see that f of x by g of x in this case is the constant c so the limit is equal to c right

so again this gives it say indeterminate form if the limit of f of x and g of x are both θ similarly if both are infinity will get indeterminate forms

so zero by zero or infinity by infinity are indeterminate forms for limits at infinity as well similarly if we have limit of f of x is infinity and limit of g of x is negative infinity then we cannot say in general what is the limit of f of x minus g of x

so infinity minus infinity this is also an indeterminate form also zero times infinity ok

so what we have discussed is we have discussed

so far limits at a point left hand and right hand limits we have seen what it means for limit to be a real number l we have also seen some examples where limits do not exist we have seen what it means for limit to be infinity or minus infinity now that infinity and minus infinity these are not number but we say limits equal to infinity if the function can be made arbitrarily large and negative infinity if it can be made arbitrarily small it can be made arbitrarily large negative number then we also saw the meaning of limit at infinity and limit at negative infinity all these concepts are very useful in determining properties of the function like to draw the graph of the function if you know the limits at infinity negative infinity and where the limit is equal to infinity or negative infinity then you can easily draw the graph of the function and also deduce certain properties about the function

so in the next lecture we will use the limits to define a very useful concepts in calculus which is the concepts of derivatives of function and we will see how to calculate derivative of function what its physical significance is and then we will see many applications of derivatives also we will use limits to define a continuous function and then discuss some properties of continuous functions thank you you