

hello everyone this is the third lecture on limits
so in the last lecture ah towards the end of the last lecture we started discussing about the infinite limits
so let me continue with that
so let me recall what do we mean by infinite limits
so definition we say that limit of f of x as x approaches a is equal to infinity if given any positive real number m there exists a positive delta such that f of x is bigger than m whenever 0 is less than $|x - a|$ is less than delta
so that is f of x can be made arbitrarily large by choosing x to be sufficiently close to a but not equal to a
so let us try to understand this with an example
so for example consider f of x is equal to $1/x^2$ and this function is not defined at x equal to 0 but it is defined for all x non-zero let us ask what is the limit
so what is limit of f of x as x approaches zero
so what we see is that $1/x^2$ if i take x to be a very small positive or a negative real number then $1/x^2$ becomes a large positive real number
so the claim is that limit of f of x as x approaches 0 this is equal to infinity and we will prove it using our definition
so so to
so this what we have to show is that given any m positive we have to find a delta
so let m greater than zero be given we need to find a delta greater than 0 such that if $|x - 0|$ is bigger than 0 and less than delta then f of x should be bigger than m now what is f of x
so but f of x bigger than m this means that $1/x^2$ we want to be bigger than m which is equivalent to writing that x^2 is less than $1/m$ and which is equivalent to $|x|$ less than $1/\sqrt{m}$
so we see that if
so therefore let us take delta to be equal to $1/\sqrt{m}$ since m is positive i can take square root of m
so this is a positive real number now if $|x|$ is less than delta and x is not equal to zero then x^2 is less than delta square which is equal to $1/m$ and this implies $1/x^2$ is greater than m since x is not equal to 0 and this is our f of x that is f of x is greater than m if $|x|$ is less than delta and x is not equal to zero
so this proves that
so hence limit of $1/x^2$ as x approaches 0 this is equal to infinity
so as we have seen that we can define the left hand and right hand limit
so similarly in definition we can define the left hand or right hand limit to be infinity which will mean that
so we say limit of x going to a minus f of x is equal to infinity or respectively limit of x going to a plus f of x which means the right hand limit is equal to infinity if given any positive real number m there exists a delta this delta might depend on m this is positive such that if we have x to be
so for the left hand limit we are only looking at the interval to the left of a
so if x is less than a and greater than $a - \delta$ this should imply that f of x is greater than m and for the right hand limit respectively if x is bigger than a and less than $a + \delta$ then f of x should be bigger than m
so for example for the function f of x equal to $1/x$ x not equal to 0 the right hand limit limit x going to zero plus of f of x is equal to infinity why is this
so proof let m bigger than 0 be given we have to find a delta positive such

that if x is greater than 0 and less than δ this should imply f of x equal to 1 by x this should be greater than m

so what should be δ here

so let δ to be equal to 1 by m since m is a real number positive real number this δ is a positive quantity and then if x is bigger than zero and less than δ which is equal to one by m then one by x is bigger than m hence the right hand limit at zero of one by x this is equal to infinity similarly we will define the limit to be negative infinity we say limit of f of x as x approaches a is equal to negative infinity if given any negative integer

so any n less than zero there exists a δ positive again δ can depend on n such that f of x should be less than the given negative number whenever zero is less than δ x minus a is less than δ

so for example

so let me write similarly we can define the left hand limit of f of x equal to minus infinity and right hand limit of f of x to be equal to negative infinity

so let us again look at the example f x equal to one by x

so we have seen that the right hand limit of this f of x at zero is equal to infinity what about the right hand what about the left hand limit

so claim is the left hand limit of f of x is equal to negative infinity this should again be a clear because if x is a negative real number then 1 by x is also negative and if you take x to be a negative real number which is very close to 0 then 1 by x will be a large negative number right but rigorously if we want to prove then let n less than zero be given then you have to find δ says that if

so let me write quickly

so take δ to be equal to minus 1 by n note that we need δ to be a positive real number

so 1 by n is negative

so minus 1 by n this is a positive real number and if n is large negative number this δ will be a small positive real number now if δ is equal to minus 1 by n then if we take

so if my x is to the left of 0 that means x is less than 0 but it is greater than 0 minus δ that means that x is greater than minus δ this implies what x is greater than minus δ which is equal to plus 1 by n okay

so if we have x to be a negative number and greater than minus δ then x is greater than 1 by n note that this 1 by n is this is negative

so this implies that minus x is less than minus 1 by n and since x is less than 0 minus x is greater than 0 and now this minus x this is positive

so this implies that 1 by minus x this should be greater than minus n and which implies that 1 by x should be less than n

so if x is less than 0 and greater than minus δ then f of x equal to 1 by x this is less than n hence the left hand limit of f of x is equal to minus infinity ok

so now let's see

so we have seen what do we mean by the limit of a function to be positive infinity or negative infinity now let's see some properties

so first suppose we have f x and g x are to function and suppose the limit of f of x is equal to infinity and the limit of g of x at a this is equal to some l which is a real number then what is limit of f x plus g x

so this again must be equal to infinity

so if we have the sum of two functions where the limit of one function is positive infinity and the limit of the other function exists and is finite then the limit of the sum has to be infinity

so to remember this

so we can write as a notation if we have any infinity plus a this is equal to infinity if a is any real number right

so whenever we have limit of the form infinity plus a real number that has to be infinity and this should be very obvious the proof is we have to show that the limit of $f(x) + g(x)$ is equal to infinity

so let m greater than zero be given then we need to find a delta positive such that $|x - a| < \delta$ and $x \neq a$ this should imply $f(x) + g(x)$ should be greater than m but we are given that the limit of $f(x)$ is infinity and limit of $g(x)$ is l

so since limit of $g(x)$ is equal to l by the definition of this limit we know that there exists some delta one positive such that $|g(x) - l| < \delta_1$ this implies that $|g(x) - l| < 1$

so note that what we are doing is here choose delta one positive corresponding to epsilon equal to 1.

so what we know is that limit of $g(x)$ as x goes to a is equal to l if given any epsilon positive i can find a delta such that whenever $|x - a| < \delta$ and $x \neq a$ then $|g(x) - l| < \epsilon$

so in particular i have chosen epsilon to be one

so $|g(x) - l| < 1$

so therefore $|x - a| < \delta$ one greater than zero this implies that $g(x)$ is between $l - 1$ and $l + 1$

so this is one we get also we are given limit of $f(x)$ as x approaches a is equal to infinity

so for any positive real number i can make $f(x)$ to be bigger than that real number for x to be

so there exists a delta 2 positive such that $f(x)$ is greater than let me write here $m + 1$ and i will write what this is later

so $f(x)$ is greater than $m + 1$ whenever $|x - a| < \delta_2$ and greater than zero

so now what happens is we have to look at $f(x) + g(x)$

so if we take delta to be minimum of delta 1 and delta 2 which is again a positive quantity since both delta 1 and delta 2 are positive then $|x - a| < \delta$ and greater than zero this will imply that $f(x) + g(x)$

so because $|x - a| < \delta$ which is less than delta 2 $f(x)$ is greater than $m + 1$ and $|x - a| < \delta$ also

so by 1 this is 1 this is 2 by one $g(x)$ is greater than we know $l - 1$

so but what we want is that we want $f(x) + g(x)$ to be greater than m where m was the given positive number

so let us write this is equal to m this is by 1 and 2.

so what should be our $m + 1$ is equal to $m + 1 - 1$

so if i choose $m + 1$ to be equal to $m + 1 - 1$ then we can make $f(x)$ to be greater than this $m + 1$ whenever $|x - a| < \delta_2$

so hence the limit of $f(x) + g(x)$ this is equal to infinity right

so let me say that this regress proof might look little bit too complicated but you should understand the idea that if the limit of $f(x)$ is infinity that means that you can make $f(x)$ to be as large as you want by choosing x to be sufficiently near a and because $g(x)$ the limit is finite

so that is near some finite number and therefore the sum can again be made as large as you want

so so similarly let's ask what happens if both the limits are infinity

so if limit of x going to a $f(x)$ is equal to infinity and limit of $g(x)$ is

also infinity then again the limit of $f(x) + g(x)$ this will be equal to infinity again the proof is let m greater than zero be given then let me write quickly

so there exist δ_1 greater than zero and a δ_2 greater than zero such that $|f(x) - a| < \delta_1$ greater than zero this imply $f(x)$ is greater than let me choose half of m $m/2$ and if $|f(x) - a| < \delta_2$ then since $g(x)$ the limit is also infinite i can make $g(x)$ to be bigger than $m/2$ for this and then by taking

so take δ equal to again minimum of δ_1 δ_2 this is positive then $|f(x) + g(x) - (a + b)| < \delta$ greater than zero this implies $f(x) + g(x)$ has to be greater than $m/2 + m/2$ which is m

so therefore the limit of the sum is again infinity here similarly you can ask suppose limit of x going to a $f(x)$ is equal to infinity then what can i say about the limit of x going to a of a constant c times $f(x)$ is it equal to infinity

so what we will see is that this is equal to infinity if c is positive this is equal to negative infinity if c is negative ok if c is 0 then c times $f(x)$ is 0 so of course this is equal to 0 if c is equal to 0 .

so why is this

so first case one if c is positive then let m greater than 0 be given then m by c is also positive because c is positive

so therefore since limit of $f(x)$ is infinity there exists δ positive such that $|f(x) - a| < \delta$ this implies that $f(x)$ is greater than this number m by c and this implies c times $f(x)$ is greater than m again because c is positive

so that means that the limit of c times $f(x)$ is equal to infinity if c is negative then we cannot write m by c that is not positive

so then minus c becomes positive and therefore if we have and less than zero is given then if i write n by c this becomes

so if i write n by minus c this is again negative because n is negative and minus c is positive

so n by minus c is a negative real number that is n by c this becomes positive so since the limit of $f(x)$ is positive infinity and n by c is a positive real number we can find a δ greater than 0 such that $|f(x) - a| < \delta$ this implies $f(x)$ is greater than n by c what i have to show is that c times $f(x)$ the limit is negative infinity that means that i have to show that c times $f(x)$ is less than n if we have $|f(x) - a| < \delta$ this is equal to minus n by minus c and since minus c is positive we can multiply both side by minus c

so minus c times $f(x)$ is again greater than minus c times minus n by minus c that is minus c $f(x)$ is greater than minus n and which is equivalent to writing c times $f(x)$ is less than n therefore by the definition limit of c times $f(x)$ is equal to minus infinity and the last case of course for c equal to zero c times $f(x)$ is equal to zero for all x where $f(x)$ is defined and

so the limit of c times $f(x)$ has to be zero right

so here you have to just be careful whether the constant is positive or negative to say whether the limit will go to positive infinity or negative infinity similarly if limit $f(x)$ is negative infinity then the limit of c times $f(x)$ this will be equal to negative infinity if c is positive this will be positive infinity if c is negative and 0 if c is 0 ok

so we have seen that if we have sum of $f(x)$ and $g(x)$ then we have some rule for the limit even if the limit is infinity but one thing we haven't seen

so what can we say about limit of $f(x) + g(x)$ if limit of $f(x)$ is infinity and limit of $g(x)$ is minus infinity

so in this case what is happening is that $f(x)$ is approaching infinity and $g(x)$ is approaching negative infinity

so it is of infinity minus infinity form

so some of you might guess that this is going to infinity this is going to minus infinity therefore infinity minus infinity is \emptyset but that is not true

so so let's see by some example

so examples one let us take $f(x)$ to be equal to one by x square and $g(x)$ equal to minus one by x square both are defined for x non-zero then here what we have is limit of $f(x)$ as x approaches \emptyset is equal to infinity and limit of $g(x)$ as x approaches \emptyset is minus infinity because $g(x)$ is nothing but minus one times $f(x)$ and by our previous thing because this minus one is negative constant if we multiply then this goes to minus infinity now what about $f(x) + g(x)$ $f(x) + g(x)$ is one by x square plus minus one by x square which is zero for all x not equal to \emptyset right

so this function is identically \emptyset in an interval containing zero

so therefore the limit of $f(x) + g(x)$ is equal to zero

so you might be tempted to think that this is always true but let us see second example let us again look at $f(x)$ equal to now i will take $f(x)$ to be equal to one plus one by x square and $g(x)$ is equal to minus one by x square and both are defined for x not equal to zero

so if we look at limit of $f(x)$ as x approaches zero this is again the limit of the first function one this goes to one and the limit of one by x square is infinity and we have seen that if we have the limit of a function is a real number and another goes to infinity then the sum is again infinity

so this is equal to infinity limit of $g(x)$ this is minus infinity but what about $f(x) + g(x)$ and here $f(x) + g(x)$ is equal to one

so x not equal to zero

so therefore the limit of $f(x) + g(x)$ is equal to one here and there is nothing special about one here if i change this one to any real number

so if instead of take $f(x)$ to be equal to one plus one by x square we take $f(x)$ equal to some constant c plus one by x square where c is any real number positive negative or \emptyset then the limit of $f(x) + g(x)$ is equal to c here right

so this infinity minus infinity it can take any real number c

so so this infinity minus infinity which does not make sense this can be any real number also

so the question is can it even be infinity or minus infinity

so the answer is again yes

so if we see consider let's see $f(x)$ and $g(x)$ says that $f(x) + g(x)$ will be equal to infinity

so if i take this is again very simple to do

so if i take $f(x)$ to be equal to two by x square and $g(x)$ to be equal to minus one by x square then limit $f(x)$ at zero is infinity limit of $g(x)$ is equal to minus infinity and $f(x) + g(x)$ is two by x square plus minus one by x square which is one by x square

so we know that one by x square approaches infinity

so limit of x going to zero $f(x) + g(x)$ this is equal to limit of one by x square x approaching zero which is equal to infinity and similarly if we take by taking $f(x)$ equal to one by x square and $g(x)$ equal to minus two by x square limit of $f(x) + g(x)$ this becomes minus infinity because here $f(x) + g(x)$ is minus one by x square limit of $f(x)$ is infinity and limit of $g(x)$ is minus infinity

so what we saw is that this infinity minus infinity can take anything

so the conclusion this infinity minus infinity is an indeterminate form which means that you cannot decide beforehand what this limit will be

so if one is going to infinity and another is minus infinity then the sum is

will depend from problem to problem what it is it's an indeterminate form and can be any real number or plus or minus infinity

so we have to carefully deal such limits similarly another thing is that ok so properties for product of functions

so suppose limit of f of x x going to some a is equal to infinity and limit of g of x is equal to some l which is a real number then what can i say about the limit of f x times g of x

so we have seen a special case of this where g of x is just a constant right

so then we saw that it depends on the sign of the constant whether the limit is equal to positive infinity minus infinity or zero

so here what we will get is that this is equal to infinity if l is positive and this is equal to negative infinity if l is negative and we will also ask what happens if l is 0

so if we have constant function 0 here then constant times f of x that is 0 in that case the limit is 0 but if the limit of g of x is zero then that is more critical

so question what if l is equal to 0

so we will answer these questions in the next lecture and we will also learn about limits at infinity in the next lecture thank you you