welcome to the second lecture on limits

so in the first lecture we gave the meaning of limits and then we also saw the regress epsilon delta definition of limits and then we were looking at some of the properties

so today i will continue with some more properties of limits and then we will state some more results

so we will continue with properties of limits

so last time we saw the sum difference rule and then the multiple of a constant

so today let us see what happens to the limit of the product of function the product rule

so this simply says that if limit of x going to a f of x and limit of x going to a g of x exists then limit of the product function f x times g x this also exists and the limit of the product is equal to the product of the limits this times limit of g of x as x goes to a

so that is limit of the product is the product of the limit again this can be proved using the epsilon delta definition but we will skip the proof rather lets see that using this let me state this result

so suppose p of x is a polynomial with real coefficients that is p of x is of the form a constant a zero plus a one x plus a two x square

so on up to a n x to the power n then the limit of p of x as x approaches any a this is equal to p of a

so to prove this you only have to know

so first note that limit of x going to a of x this is equal to a right this is very easy to c n using epsilon delta definition you just have to see that delta equal to epsilon will work

so the limit of the function f x equal to x the limit at x goes to a is equal to a

so using this

so therefore the limit of x square as x goes to a this will be equal to simply a square by the product rule and by induction limit of x to the k x going to a this is equal to a to the k for all k in natural number

so therefore the limit of p of x as x goes to a is equal to first of all this i can write as limit of a 0 plus limit of a 1 times x and

so on up to limit of a n times x to the n as x goes to a this is by the sum rule and then you use that limit of a constant time a function is constant time the limit

so this implies limit of p of x is x goes to a is equal to limit of the constant is simply a zero plus a one times the limit of x as x goes to a plus a two times limit of x square x going to a

so on up to a n times limit x to the n is x goes to a this is equal to a zero plus a one the limit of this is simply a plus a two times a square and the limit of x to the n is a to the n but this is nothing but the value of the polynomial evaluated at a p of a

so hence we are done

so for example if we want limit as x going to say one of x square plus 3 x plus 2 then you can just write this as the value of the polynomial at 1

so that is 1 square plus three times one plus two which is one plus three plus two which is six right next rule is about the quotient of two functions

so suppose limit of x going to a f x exists and limit x going to a g of x this also exists further and is not equal to zero then the limit of the quotient of these two functions f x by g of x this is equal to limit of f of x by limit of g of x right

so this is the result but if the limit of the denominator g of x if that is equal to 0 then division by 0 is not defined

so of course we cannot write this but if the limit is non-zero then this makes sense and we can use this formula that limit of cosine is the quotient of the limit provided the limit of the denominator is not equal to zero
 so this means that that is the limit of the quotient of two functions equals the quotient of the limit if the limit of the denominator is non zero
 so for example if we have to calculate limit of say x cube plus two x plus three divided by x square plus one as x approaches one
 so here first you see that what is the limit of the denominator as x approaches one x square plus one is polynomial
 so the limit of x square plus one is one square plus one which is non-zero
 so this i can write that since limit of x going to one x square plus one this is equal to one square plus one which is two this is non zero therefore the limit of x going to one x cube plus two x plus three by x square plus one is simply the quotient of the limits and now both are polynomials
 so we know that we just need to evaluate it
 so at x equal to one we get one cube plus two plus three this is six divided by two
 so this is three let me warn you we said that if the limit of the denominator function is non-zero then the limit of the a quotient exists and equals to the cosine of the limit but if the denominator function limit is 0 we cannot say that the limit does not exist
 so the limit of f x by g x this might exist even when limit of g of x is equal to zero in fact we will see that most of the important examples will be when the limit of the denominator is actually equal to zero
 so for example find limit of x going to one x square minus three x plus two divided by x minus one if it exists
 so here we see that the limit of x going to one of x minus one this is simply one minus one which is zero also
 so we cannot use the directly the quotient rule
 so we cannot use the quotient rule directly but if we see the numerator limit of numerator that is again 1 square minus 3 times 1 plus 2 which is also 0.

 so the limit of the numerator and denominator both are zero in fact here we see that at x equal to one the numerator and denominator both are zero
 so therefore x minus 1 is a factor of
 so here x square minus 3 x plus 2 divided by x minus 1 this is equal to x minus 1 times x minus 2 divided by x minus 1 of course this is defined if x is not equal to one and if x is not equal to one then one can cancel this x minus one and this is equal to x minus two if x is not equal to one
 so the limit of x tends to one of x square minus three x plus two by x minus 1 this is same thing as limit of x approaching 1 of x minus 2 because remember that while calculating the limit we do not have to consider the value of the function at x equal to a we only need to know when x is sufficiently close to one
 so this is equal to again this is just a polynomial
 so this is 1 minus 2 which is minus 1 right
 so in this example we see that even though the limit of the denominator does not exist the limit of the quotient might exist right
 so this is one way of trying to find the limit let us do one more example
 so what is limit as x approaches four of x minus four divided by square root x minus two
 so here the function is
 so here we have this function f of x equal to x minus 4 divided by square root x minus two this is defined for any x which is greater than equal to zero but x

not equal to four right
 so if i take an interval and any in interval which ah any interval around four which does not contain negative integers negative real numbers then this function is defined and we want to know what is the limit
 so we can talk about the limit of this function
 so note that this is again we can simplify
 so we can multiply and divide by square root x plus two this is the conjugate of square root x minus two to get f x is equal to x minus 4 times square root x plus 2 by square root x minus 2 times square root x plus 2 and now if you multiply in the denominator we get x minus 4 and this cancels with this
 so f x is equal to square root x plus 2 if x is greater than equal to 0 and x not equal to 4.

 so therefore limit of f of x as x approaches 4 is equal to limit of this function square root x plus 2 which is same as the function f of x in a small enough interval containing four excluding x equal to four
 so this is equal to square root four plus two which is two plus two which is four
 so this is another way of calculating limits when you get the limit of the numerator and denominators are both zero
 so next what will do is this theorem this is called this sandwich theorem or sometime it is called the squeeze theorem
 so what this says is that suppose we have a function f of x suppose f of x is greater than equal to g of x and is less than equal to h of x for all x in an interval containing a but may be excluding x equal to a
 so suppose f x lies between these two function g of x and h of x in an interval containing a also assume that the limit of g of x equals the limit of h of x and both are equal to say l
 so if the limit of the function g of x and h of x are both equal to l then the conclusion is that then limit of f of x as x approaches a this exists and it equals again the same limit l right
 so if you see using a graph
 so what this says is that suppose you have a function says that in some interval this f x is between two functions g of x and h of x and the limit is same right
 so in this case let me draw this
 so we have f of x and we have the g of x and h of x
 so in in case that the limit at this of the upper function h of x and the lower function g of x if they are both same then the limit of f of x is also same
 so remark the sandwich theorem can be proved using the epsilon delta definition
 so i would suggest that the students can try to prove this using this definition but we will skip the proof rather they say that this is very important
 so this theorem is is extremely useful in computing limits
 so many times what happens is that the function whose limit you want to calculate it might be complicated but if you can find a smaller function and a bigger function and you can easily calculate the limits and if the limits are same then this also has the same limit
 so we will see an important limit using this theorem
 so as an application we will prove that limit of the function sign x over x as x approaches zero this is equal to one
 so this is a very important formula for limit that limit of sin x by x at zero is equal to one note that here we cannot use note that we cannot use the quotient rule because limit of the denominator is 0 limit of the numerator is

also 0 and here we do not have polynomials that we cancel x and then try to use the quotient rule

so rather what we will do is that we will prove that this limit is equal to 1 by using sandwich theorem

so to do that let me first draw let me draw a circle of radius one

so i have a circle and the radius of this circle is one

so i have this this is one this is one and now let us take an angle equal to x radian

so this is x in radian and let me draw this triangle this and let me join these two points and we will also extend this and then draw a perpendicular here

so let me mark these points

so i have o here a this point is b this is c and this is d consider a circle of radius one and let x be an angle in radian and consider the figure drawn to the left now let us try to

so from this figure it is clear that if i look at the area

so area of triangle o a b this is less than area of sector sector oab which is in terms less than area of the bigger triangle o a d right

so we have this is the sector this is a fraction of the whole circle and of course the triangle oab this area is less than the area of this sector and this sector is again contained strictly between this triangle o a d

so we get this right now let us see what is the what are these areas we have this now area of triangle o a b is equal to half times the base is o a times b c right which is equal to half the length of oa is one this is one what is the length bc

so note that this angle is x

so this length bc is nothing but sine sine x right sign of x is nothing but the ratio of the opposite side and the hypotenuse which is one here

so sine of x is bc divided by one

so bc is equal to sine x

so this is half times one times sin x

so we get area of triangle o a b is equal to half sine x now what is the area of the sector area of sector o a b

so the area of the sector is in the ratio of the area of the is in the ratio of the angle

so we have angle is x divided by the total angle of the circle is two pi radian

so x divided by two pi into area of the circle right for two pi angle we get the whole area of the circle

so for x angle we get x divided by two pi into area of the circle

so this is equal to x divided by two pi area of the circle is nothing but pi times the radius one square

so this gives me just half x

so area of sector is half x and we also need area of triangle o a d

so this is again a right angle triangle and the area is half times o a times a d let me show the picture again

so the triangle o a d the area is half times o a times a d what is the length of a d in this triangle o a d if we see this base is of length one

so this opposite side is of length tan x

so this is equal to half times o a times a d

so that is half times one times tan x this is half tan x

so what we get is that

so for any x between zero to pi by two we have that area of triangle o a b is half sine x is less than area of the sector which is half x which is less than half times tan x right this is true for every angle x which is between zero and pi by two now what we do is that we divide throughout by by sign x if x is

between zero and pi by two sin x is positive
so dividing by this what we get is
so first let me cancel this half from each of this
so we get sine x is less than x is less than tan x and then dividing by sine x
i get one is less than x by sine x which is less than tan x divided by sine x
but tan x is sine x divided by cos x
so this is equal to one by cos x
so the conclusion is what we get is
so one is less than x by sine x which is less than one by cosine of x for every
x between zero and pi by two now if we take the reciprocal this implies that if
i take the reciprocal the inequality changes
so we get 1 is greater than sine x by x which is greater than cos x for all 0
less than x less than pi by 2 right
so this is a very important inequality
so we have got that let me write this again cos x is less than sine x by x
which is less than one if zero is less than x is less than pi by two right what
we were looking for is the limit of sin x by x what we did is we have found that
sin x by x is between cos x and one for all x between zero and pi by two
so also if i replace x by minus x we note that cos of minus x is nothing but
cos of x and sine of minus x divided by minus x because sin is an odd function
this is minus sin x divided by minus x which is again equal to sin x by x that
is both cos x and sin x by x are even functions
so if x is negative then minus x is positive
so because we have this inequality one
so hence from one the inequality is in fact true for any x between minus pi by
two to pi by two right
so now we got an interval containing zero in which we have this inequality that
sin x by x is greater than cos x and less than one now we just need to know what
is the limit of cos x and limit of 1 as x approaches 0 since limit of cos of x
as x approaches 0 this is equal to simply cos of 0 which is 1 and of course the
limit of the constant function one is equal to one by the sandwich theorem the
limit sign x by x as x approaches zero this is also equal to one
so we have proved that this limit is equal to one by using by bonding this
function between two functions cos x and one whose limits are easy to calculate
and then we know that this limit is also equal to 1.

so this is a very important formula for limit and you can use this to calculate
many limits
so we will do some examples using this formula
so one is if i write limit of tan x by x this is equal to what
so note that tan x by x this is equal to sine x by x times 1 by cos x if x is
not equal to 0 and we know that limit of sin x by x this is equal to 1 and limit
of one by cos x this is equal to one by limit of cos x is one
so this is also one
so by the product rule limit of tan x by x this is also equal to one another
one important limit is what is this limit of one minus cos x divided by x
so here we have that as x goes to 0 the limit of the denominator is 0 the limit
of the numerator is again 1 minus 1 which is 0 and we want to calculate this
limit
so recall from trigonometry we can express cos of an angle in terms of sine of
half of the angle right
so cos of 2a is equal to 1 minus 2 sine square a right cos of 2a is cos square
a minus sin square a which is equal to one minus two sin square a
so that is one minus cos two a is equal to two sine square a

so therefore one minus cos x is nothing but two times sine square of x by two
so the numerator is equal to two times sin square x by two denominator is x
so one minus cos x by x this is equal to two sine square x by two divided by x if x is not equal to zero and then we can write this this is equal to sine of x by two divided by x by two times another sign of x by 2 right i just brought this 2 in the denominator
so i get x by 2 here
so therefore limit of x going to zero of one minus cos x by x this is equal to limit of x going to zero of sine x by two divided by x by two times the limit of sine of x by two as x goes to zero provided these two limits exist right
so now what we know is that limit of sine x by x as x approaches zero that is equal to one here what we have is sine of x by two divided by x by two
so if we put y is equal to x by two then as x tends to zero y also tends to zero because y is half of x
so therefore this limit x going to zero of sine x by two divided by x by two is equal to limit of sine y by y where y is approaching zero and this limit we know is one and of course the other limit limit of x going to 0 of sine x by 2 this is equal to sine of 0 which is 1 which is zero
so therefore limit of one minus cos x by x x going to zero this is equal to zero
so we saw three limits in terms of trigonometric functions one is sine x by x limit as zero is one and using that limit of tan x by x at zero is also one and limit of one minus cos x by x that is equal to zero
so some other example using these formulas
so what is limit of a tan of two x divided by sine three x
so what we can do is that this is equal to we can write tan two x divided by two x and then multiply it by two x if x is not zero then i can write tan two x as tan two x by two x times two x similarly sine three x can be written as sine three x by three x times three x and then we have limit as x goes to zero now in this one two x by three x if x is non zero i can cancel this x and then we know the limit of tan y by y as y approaches zero that is one limit of any sign y by y as y approaches zero that is also one
so this is equal to the constant can be taken out from the limit
so two third limit of x going to zero tan two x divided by two x and then divided by limit sine three x divided by three x which is equal to two third we have seen that both the limits are one
so this is equal to two-thirds another example find limit of x going to zero of sin three x divided by x
so many students make mistakes thinking that this sign of three x this is going to zero x is also going to zero and we have seen that the limit of sin x by x that is equal to 1 but this limit is not equal to 1 because here what we have to do is that we need to use the same thing
so we write limit x going to zero sin three x divided by if i write three x then here i have whatever is inside sign i am dividing by that this limit is equal to 1 and but then i have to multiply this by 3 to get the same function
so this limit is equal to this limit and now because i am dividing by the same 3x which is inside sine 3x then i can simply write or if you want you can write one more step this is equal to limit of y going to zero sine of y by y times three and where y is equal to three x and then this is equal to three
so so it is a good idea to like change it into another variable y and then they use the formula for sine y by y right
so at least in the beginning you should do like this to avoid the mistake that this is not equal to limit sine three x by x x going to zero this is equal to one this is wrong right

so the next thing that we will talk about is what we call infinite limits

so i i will just explain by an example

so suppose you look at consider f x equal to one by x for x not equal to zero

so this function if you see as we approach 0 as x becomes a small positive number 1 by x becomes larger and larger

so the graph of this functions is this rectangular hyperbola and for negative x we have this as x is negative

so what what happens is that as x tends to 0 plus this means from the right 1 by x becomes larger and larger and as x tends to zero minus that means that from the left if you approach zero then one by x becomes large negative number

so we will explain this more rigorously but what we say

so in such cases we say limit of 1 by x as x approaches to 0 from the right this is equal to positive infinity and limit of x going to 0 minus of 1 by x this is equal to negative infinity

so this simply means that if x is a small positive number then 1 by x becomes a large positive number and if x is a small negative number then 1 by x becomes a large negative number

so we use this symbols positive infinity and negative infinity to just say that in in the next class i will make it more rigorous and define when we say that the limit at a point is positive infinity or negative infinity and then we will see some more examples

so in the next class we will make it more regress about this limit being positive infinity and negative infinity and then we will see some more examples on this thank you you