

hello everyone

so today is the first lecture on limits

so today i will explain what is what do we mean by limits

so limits are very important concepts and it is the backbone of whole of calculus

so this is very important chapter for you

so let me start with the definition of limit

so suppose f is a function defined on an interval containing x equal to a except possibly at x equal to a

so we would like to define what do we mean by the limit of the function f of x as x tends to a

so the notation for this is notation we will use is for the limit we write $\lim_{x \rightarrow a} f(x)$ limit as x tends to a of the function f of x right this denotes the limit of f of x as x tends to a

so informally what it means is formally the limit $\lim_{x \rightarrow a} f(x)$ is the real number l such that $f(x)$ is arbitrarily close to l if x is chosen to be sufficiently close to a of course excluding the point x equal to a

so let me explain what we mean by this arbitrarily close to l and sufficiently close to a

so suppose i have a function and we have x equal to a here and this function is something like this it might not be defined at a

so if we look at this definition the limit of $f(x)$ as x tends to a this is this number l such that if you want the value of $f(x)$ to be as close to l as you want

so suppose i want my $f(x)$ to be in this interval containing this point l then from this picture you can see that if i choose my x to lie in this interval then if x belongs here x lies in this interval then $f(x)$ lies in this interval let me call this interval as I lies in the interval I

so in case there is no such l then we say that limit $f(x)$ as x tends to a this does not exist and if there is such an l then we will say that the limit is equal to l

so for example if i look at this function

so let me write example one suppose i have this function $f(x)$ which is equal to zero for x less than 0 and it is equal to 1 for x bigger than 0

so this is function $f(x)$ equal to zero if x is less than zero and this is one if x is greater than equal to zero

so now here if you see

so does limit $f(x)$ as x approaches 0 exists

so in this case if you see if i take my x to b in any interval containing 0 then for positive x we know that the value of $f(x)$ is 1 and for any negative x no matter how small it is we have this is equal to zero

so in this case the answer is no because for x greater than zero $f(x)$ is equal to one and for x less than zero $f(x)$ is always zero

so no value of l will satisfy our requirement

so there is also concept of one sided limits

so in this case let me define one sided limits

so there are two one is called left hand limit and the second one is right hand limit

so what is left hand limit

so this will use the notation $\lim_{x \rightarrow a^-} f(x)$ as x tends to a minus this means that here we are looking at the function

so let me again explain by this example

so the function is 1 for x greater than 0 greater than equal to zero and zero for x less than zero

so in this case for $f(x)$ equal to zero for x less than zero one for x greater

than equal to zero the left hand limit $\lim_{x \rightarrow 0^-} f(x)$ is equal to zero this is because if I take any point in this interval to the left of this point then $f(x)$ is identically equal to 0

so left hand limit let me write

so $\lim_{x \rightarrow a^-} f(x) = l$ if we can make $f(x)$ to be arbitrarily close to l by choosing x to lie in the interval $(a - \delta, a)$ for some $\delta > 0$ sufficiently small. Similarly we can define right hand limit this is denoted by $\lim_{x \rightarrow a^+} f(x) = l$ if $f(x)$ can be made as close to l as we want by choosing x to lie in a small enough interval of the form $(a, a + \delta)$

so this is called right hand limit because we are only considering the value of the function in the interval to the right of a whereas for left hand limit we are interested in values of x in an interval to the left of a

so note that for the left hand limit only the values of the function to the left of a is important and similarly for the right hand limit only the values of $f(x)$ to the right of a is important also the value of $f(a)$ is not at all important for the purpose of knowing the limits whether it is $\lim_{x \rightarrow a} f(x)$ or left hand limit of $f(x)$ at a or the right hand limit of $f(x)$ at a

so another remark is that the limit of $f(x)$ at a as x approaches a this is equal to l if and only if both the left hand limit and the right hand limit exist at $x = a$ and are equal to l

so the limit exist if and only if left hand limit and right hand limit both exist and both are equal to l

so for the example $f(x) = 0$ for $x < 0$ and 1 for $x > 0$ if we calculate $\lim_{x \rightarrow 0^-} f(x)$ the left hand limit this is clearly equal to zero and the limit of $x \rightarrow 0^+$ $f(x)$ this is equal to one

so both left hand limit and right hand limit exist at zero but since they are not equal the limit $\lim_{x \rightarrow 0} f(x)$ this does not exist ok

so can we have the case that even the left hand or right hand limit does not exist

so in the previous example we saw that left hand limit and right hand limit exist but they are not equal

so the limit does not exist but can we have this case

so let us consider this let $f(x) = \sin(1/x)$ for $x \neq 0$

so does $\lim_{x \rightarrow 0} \sin(1/x)$ exist

so we are defining $f(x) = \sin(1/x)$ for all $x \neq 0$ and $f(x)$ is defined to be equal to $\sin(1/x)$

so if you draw the graph of this function

so note that $\sin(1/x) = 0$ for all x of the form $x = 1/n\pi$ where n is a natural number right we know that \sin is 0 at all integer multiple of π

so if I take $x = 1/n\pi$ then $\sin(1/x) = \sin(n\pi) = 0$ right also if $x = 1/(2n + 1)\pi$ then $\sin(1/x) = \sin((2n + 1)\pi) = 0$ right

so if n is even we get one if n is odd then we get minus one in fact if x belongs to the interval say

so if I take x to be in the interval say $(1/(2n + 1)\pi, 1/2n\pi)$ then that is x is in the interval $(1/(2n + 1)\pi, 1/2n\pi)$ then $f(x) = \sin(1/x)$ takes all values between zero and one right

because in the interval $2n\pi$ to $2n\pi + \pi$ sine of x takes the value from 0 to 1 in fact if i take from $2n\pi - \pi$ to $2n\pi + \pi$ then it takes all the values from minus one to one

so what we see is that if you would we try to draw the graph of this x and then i have $f(x)$ equal to sine of x

so if i am interested in the right hand limit i need to look at the value of this function to the right of this

so if you see if i take some one by $2n\pi$ and then i have one by $2n\pi + \pi$ then the value of this is varying from i have zero in lets say this is one

so this takes all the values from zero to one and then again at one by $2n\pi + \pi$ it becomes zero again and then it again goes to minus one then it again goes to one

so as you go near x equal to zero this keeps oscillating

so what happens the function $f(x)$ keeps oscillating between minus one and one in any interval containing zero thus if i am looking at right hand limit of this function this does not exist similarly the left hand limit also does not exist for this function

so now

so far today i have given some intuitive notion of the limit of a function now next i try to give the regress definition

so this is also called the epsilon delta definition of limit

so ok

so the definition is we say that limit of $f(x)$ as x tends to a this exists and is equal to l if given any epsilon positive

so if you take any real number epsilon which is positive then there exists a real number delta which is also positive such that if i take x to be line such that if $|x - a| < \delta$ and greater than zero then $|f(x) - l| < \epsilon$ let me highlight these things

so let me explain this again by graph

so we have x and y is equal to $f(x)$ and we have some function now we have this l

so as we explained before that the limit is that value if it exists such that if i approach this x equal to a from either left side or right side then the value of the function becomes arbitrarily close to this number l

so suppose i want that my $f(x)$ should be in between $l - \epsilon$ to $l + \epsilon$ then i can choose a delta such that if my x is in this open interval $a - \delta$ to $a + \delta$ except possibly at x equal to a then my $f(x)$ should lie in this interval $l - \epsilon$ to $l + \epsilon$ for example you might want that this epsilon is equal to say 0.1

so if you take epsilon equal to point one maybe there is some small enough interval here $a - \delta$ to $a + \delta$ for which the function value is between $l - 0.1$ to $l + 0.1$ now if you make this epsilon smaller say point zero one then still there is another delta presumably smaller than the previous one such that if i choose here then again this is between $l - 0.01$ to $l + 0.01$

so this is the definition of the limit

so remark delta generally depends on epsilon

so if you make epsilon is smaller then delta might have to be made smaller

so one example consider the function $f(x)$ equal to say x^2 and then what is limit of $f(x)$ as x approaches 0

so if you see this function this is very simple this is a parabola

so here if you see the left hand limit if you approach this function from the left side then it is approaching this point which is zero and from the right again if you approach then this is becoming closer and closer to zero

so intuitively from the graph it is clear that the limit is zero let us try to prove this using the epsilon delta definition in this what you have to do is if you are given an epsilon then you have to tell what is delta such that the condition is satisfied

so we start with let epsilon be any positive number what we want we want a delta positive such that if $|x - a| < \delta$ then $|f(x) - L| < \epsilon$ let me write

so $x - a$ is δ here if this is less than delta and greater than 0 this should imply that $|f(x) - L| < \epsilon$ we are claiming that the limit is 0 this should be less than epsilon that is if $|x - a| < \delta$ and greater than zero this should imply that $|x^2 - a^2| < \epsilon$ now we see that now if $|x - a| < \delta$ then $|x^2 - a^2| < \delta^2$ this has to be less than epsilon

so if we want $|x^2 - a^2| < \epsilon$

so if we choose delta equal to square root of epsilon then $|x - a| < \delta$ implies $|x^2 - a^2| < \delta^2$ which is equal to epsilon and delta is of course positive if epsilon is positive

so from the definition from the epsilon delta definition it follows that limit of x^2 as x goes to a this has to be zero

so this was very simple example but it was just to illustrate how we use this epsilon delta definition to prove that limit is equal to some number okay next what we will do is we will see some properties of limit

so some properties of limits

so the first one is this sum rule

so this says that suppose $f(x)$ and $g(x)$ are two functions such that $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} g(x)$ this also exists then the conclusion is then $\lim_{x \rightarrow a} (f(x) + g(x))$ this must exist and the limit is equal to $\lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$ at $x = a$ is equal to the sum of the limit of $f(x)$ and $g(x)$

so briefly it says that the limit of sum is equal to sum of the limits limit of sum is equal to sum of the limits provided each limit exists

so this again is a something which is not very hard to believe that limit of the sum is sum of the limit but if you try to prove this then this epsilon delta definition comes handy

so proof

so let $\lim_{x \rightarrow a} f(x) = l_1$ and $\lim_{x \rightarrow a} g(x) = l_2$

so by using the definition what it means is that and to

so what we want to show is that $\lim_{x \rightarrow a} (f(x) + g(x)) = l_1 + l_2$

so to do to show this we need to find a delta

so let epsilon greater than zero be given we need to find a delta

so first of all we are given the limits of $f(x)$ and $g(x)$

so since $\lim_{x \rightarrow a} f(x) = l_1$ we know that there exists some delta one positive such that if $|x - a| < \delta_1$ and greater than 0 then this should imply that $|f(x) - l_1| < \epsilon/2$

so instead of epsilon i will put here epsilon by two because what it says is that given any epsilon you can find a delta such that this happen

so it should be true for epsilon by two also

so this is my equation one similarly i can find there exists a delta two positive such that $|x - a| < \delta_2$ and $x \neq a$

this should imply that $|g(x) - l_2|$ is less than $\frac{\epsilon}{2}$ again. Let's call this two.

so now what we have to show is we have to find a δ such that

so if I take now we have the function is $f(x) + g(x)$ and we have to prove that the limit is $l_1 + l_2$.

so if we take $f(x) + g(x) - (l_1 + l_2)$ and we have to make the difference in absolute value to be less than ϵ .

so this is equal to $|f(x) - l_1 + g(x) - l_2|$ and then we know that $|a + b| \leq |a| + |b|$.

so this is less than or equal to $|f(x) - l_1| + |g(x) - l_2|$. This is because $|a + b| \leq |a| + |b|$ is always true. Now what we are given from one is $|f(x) - l_1| < \frac{\epsilon}{2}$ if $|x - a| < \delta_1$ and this is less than $\frac{\epsilon}{2}$ if $|x - a| < \delta_1$ and also $|g(x) - l_2| < \frac{\epsilon}{2}$ if $|x - a| < \delta_2$. This is from one and two and this sum is equal to ϵ .

so if I choose δ to be equal to the minimum of δ_1 and δ_2 then if $|x - a| < \delta$ and less than δ then because δ is the minimum $|x - a| < \delta_1$ as well as $|x - a| < \delta_2$. This will imply that $|f(x) + g(x) - (l_1 + l_2)| < \epsilon$. Thus by definition limit of $f(x) + g(x)$ is equal to $l_1 + l_2$. ok.

so this rule will be very important because very useful because if you know the individual limits then you can find limit of the sum of functions. Right? Second property: limit of $\alpha f(x)$ as $x \rightarrow a$ is equal to α times limit of $f(x)$ as $x \rightarrow a$ where α is any real number and this is true if the limit of $f(x)$ as $x \rightarrow a$ exists. Right?

so if I multiply the function by any real number then the limit of this constant times $f(x)$ is equal to the constant times limit of $f(x)$.

so that is limit of a constant times $f(x)$ equals the constant times the limit of $f(x)$. Right?

so proof will not do proof I leave it as an exercise.

so again using the ϵ - δ definition you should try to find what is the δ which will work for this. ok.

so and the third one is of course just like the sum rule. If I take difference: limit of $f(x) - g(x)$ is equal to limit of $f(x)$ minus limit of $g(x)$ provided the limits on the right hand side exist. This is just like the sum rule. If limit of $f(x)$ and limit of $g(x)$ exist then the limit of the difference is equal to difference of the limit and this in fact follows from those from property one and two of course one can prove directly also using ϵ - δ definition but let me show that this actually follows from one and two.

so we have $f(x) - g(x)$ you simply write this as $f(x) + (-1)g(x)$. Right?

so now we have sum of two functions $f(x)$ and $(-1)g(x)$.

so since this is true limit of $f(x) - g(x)$ is equal to limit of $f(x)$ plus limit of $(-1)g(x)$. This is by property 1 (sum rule) and then by the second property limit of $(-1)g(x)$ is (-1) times limit of $g(x)$.

so that is limit of $f(x)$ minus limit of $g(x)$. ok.

so this brings to the end of first lecture. In the second lecture I will show some more properties of limits and then we will also do some more examples and calculate some limits. Thank you.