hello everyone we will continue our discussion on conic sections in particular we will study about ellipse and hyperbola in this lecture and the next

 so in the last lecture we discussed about equation of tangent normal etcetera to an ellipse this lecture will continue first with ellipse

 so first thing we will derive is formula for equation of chord of contact of tangents from an external point x one y one to an ellipse

 so consider the ellipse x square by a square plus y square by b square equal to one and let x one y one be a point outside the ellipse

 so we have this ellipse and we have some point x one y one which lies outside the ellipse now from this point x one y one two tangents can be drawn to the ellipse and what we are required is to find the chord of contact

 so let p be this point and q and r with the points on the ellipse where the tangents are drawn and then we want to find the equation of this chord q r

 so let this is say point p suppose p q and p are are the two tangents where q and r lie on the ellipse

 so let q be the point x prime y prime and r with the point x double prime y double prime on the ellipse then the equation of the tangent line

 so recall equation of tangent at some point x prime y prime is given by x x prime by a square plus y y prime by b square equal to one

 so therefore this in this case will be the equation of line p q and similarly equation of line p r is x x double dash by a square plus y y double dash by b square equal to one

 so these are the equation of the line p q and p are now since the point p x one y one lies on both p q and p are we have

 so if we put x one y one in this equation for p q and we get x dash x one by a square plus y dash y one by b square equal to one and x doubled as x one by a square plus y doubled as y one by b square this is also equal to one

 so we get these two equations one and two what we had to find we have to find the equation of the line q r

 so we need equation of the chord q r

 so notice that q and r these are the points x dash y dash and x double dash y double as

 so if we have any straight line on which these two points lie then that will be the equation of the straight line q r

 so consider the equation x times x one by a square plus y times y one by b square equal to one this is equation of a straight line note that one and two implies one implies point q which is x dash y dash lies on the above line and two implies the point r whose coordinates are x doubled as y double dash this also lies on the same line now there is a unique straight line passing through any two points therefore this equation is nothing but equation of the line joining q and r

 so since there is a unique straight line passing through two distinct points the equation is of the line joining q and r thus what we get is that equation of q r is x x one by a square plus y y one by b square equal to one

 so note that this equation looks very similar to the equation of the tangent line at a point x one y one but in this case the point x one y one lies outside the ellipse

 so we have this ellipse and we have any point x one y one and then the equation of the chord joining this two point q and r is nothing but plugging in x one y one in the equation of the tangent line at a point on the ellipse this gives us the formula for the equation of the chord next we will do this problem prove that the normal at any point on the ellipse bisects the angle between the lines to the foci

 so lets draw a diagram we have an ellipse and suppose there is a point p

so we have there are two foci lets call them f one and f two and suppose we take any point p on the ellipse and then we look at the line joining the focus foci to this point p what we have to prove is that the normal at this point p this is the normal line this bisects the angle this

so to show the normal at p bisects angle f one p f two

so what we will do is let us look at this ellipse and lets look at the point p and the two foci f one and f two and here is ok

so let us look at the angle bisector of this

so let p n be the angle bisector of angle f one p f two and what we have to show is that we need to show that this is the normal

so when need to

so that p n is normal to the ellipse

so since we have taken this angle bisector let us call these angles to be theta now let us say let us look at the line which is perpendicular to this p n then this angle is pi by two minus theta this is also pi by two minus theta

so con consider the line passing through p which is perpendicular to p n

so proving that p n is normal is equivalent to proving that this line is tangent

so will show that this line is tangent to the ellipse which means that we have to show that if there is any other point q on this line then q must not lie on the ellipse

so take a point q on this perpendicular line let me draw the picture again this is the point p this is n this is f one and f two now suppose the equation of the ellipses x square by a square plus y square by b square equal to one then consider a point lying on this line f to p which is at distance two way from the focus f two

so this distance p l is two times a

so consider the point l on the line f 2 p such that the distance f two l this is equal to two a now we take any point q on this line we have to prove that this q lies outside the ellipse

so this distance is two a what is the

so now notice this two a is equal to distance of l two f two which is less than the distance of q f two plus q l

so if i join this point q to f two and l this is just this sum of two sides of a triangle is greater than the third side

so q f two plus q l is strictly bigger than the distance l f 2 which is 2 a but what can i say about ok let us look at this q f one

so note that since p n bisects angle f one p f two we get that p q bisects angle f one p l this is because this is perpendicular lines

so if you want to see this if this angle is theta and this angle is also theta

so this angle is pi by 2 minus theta this is also pi by 2 minus theta and therefore this angle is again pi by two minus theta these two angles are equal

so we see that p q is the bisector of angle f one p l and therefore these two triangles are congruent and therefore q l is equal to q f one

so therefore ql is equal to q f one

so what does this imply this implies two a is strictly less than q f two plus q f one now recall that the ellipse is such that for any point on the ellipse the sum of the distance from the two foci is equal to two a because here the sum of the distance is greater than two a this implies q lies outside the ellipse and when hence we have proven that this p n is normal to the ellipse ok next problem need to find the locus of the foot of perpendicular from the foci of an ellipse to the tangents

so suppose we have consider the ellipse in the standard form x square by a square plus y square by b square equal to one with a greater than b

so we have this ellipse now what we have required to do suppose i take any general tangent to the ellipse and then if you look at the to foci f1 and f2 you can draw perpendicular to the tangent from the foci f1 and f2 then we have to find the locus of all such points

so we know that the equation of tangent

so recall that the equation of tangent whose slope is m is given by y equal to m x plus c where c square is a square m square plus b square

so for any fixed value of m there are two tangents that is y is equal to m x plus or minus square root of a square m square plus b square this gives us the equation of the tangent for different values of m now if we have to find suppose this is any general point h k is the foot of perpendicular then the equation

so if h comma k is the foot of perpendicular from foci plus minus ae zero then this slope of line joining h k two foci is minus one by m because this is perpendicular to the tangent line and therefore the equation would be k minus zero equal to minus one by m times x minus or plus a e this implies m k plus x x is h sorry m k plus h square is equal to a square e square this is one equation also h comma k lies on the tangent line therefore where equation of tangent line is this

so k minus m h square is equal to a square m square plus b square this is equation two if we add one and two then we get m square k square plus h square plus two m h k plus k square plus m square h square minus two m h k equal to a square e square plus a square m square plus b square

so we see that this cancels here this will imply one plus m square is common and we get h square plus k square equal to a square what is e square e square is 1 minus b square by a square plus a square m square plus b square

so this is equal to a square minus b square plus a square m square plus b square b square cancels and this is equal to a square times one plus m square

so one plus m square can be cancelled and this gives h square plus k square equal to a square since this is true for any h comma k which is foot of the perpendicular therefore the locus is x square plus y square equal to a square

so in this picture this h comma k lies on circle of radius a centered at the center of the ellipse

so this this is called auxiliary circle

so the foot of the the locus of the foot of perpendicular from foci to tangent is the auxiliary circle of the ellipse ok next problem we want to determine what is the maximum possible area of a triangle inscribed in an ellipse

so to solve this problem let us use the parameter for a general point on the ellipse

so let's a equal to

so a cos theta b sin theta b equal to a cos phi b sine phi c equal to a cos psi b sin psi b any three points on the ellipse x square by a square plus y square by b square equal to one

so we are considering any three points on the ellipse where theta phi and psi are the accent triangles

so that is theta phi psi are the eccentric angles of the points

so we saw in the last class that if we have this ellipse and if we take any point here a b and c then this eccentric angle is nothing but if i look at the auxiliary circle then we look at the corresponding points on the auxiliary circle

so let us look at this point p this point q and corresponding to c there is a point this is c and this is point is r

so this theta phi and psi are actually the angle of the point p

so p is the point a cos theta b sin theta sorry a cos theta a sin theta

so let me write it in the next space

so let p equal to a cos theta a sin theta q equal to a cos phi a sin phi now r equal to a cos psi a sin psi b the corresponding points on the auxiliary circle x square plus y square equal to a square now recall if we have the coordinates of any three points as x one y one x two y two and x three y three then area of triangle whose vertices are x one y one x two y two x three y three is half of we look at determinant of x one y one one x two y two one and x three y three one and the absolute value of this gives the area of the triangle

so we will use this formula therefore area of triangle a abc is equal to half times look at the determinant of x one is a cos theta x 2 is y 1 is b sine theta 1 a cos phi b sin phi 1 and a cos psi b sin psi one this is equal to half a is common from the first column and b is common from the second column

so half a b times cos theta sin theta 1 cos phi sine phi 1 and cos psi sin psi 1.

also what is area of triangle p q r where now p q r are points on the auxiliary circle of radius a

so here the difference is only the y coordinates are instead of b sin theta i have a sin theta similarly a cos phi a sin phi one a cos psi a sin psi n one this is equal to half a square times cos theta sin theta one cos phi sine phi one and cos psi sin psi one

so this is same in here

so what we see is that therefore area of triangle abc by area of triangle pqr is equal to if i divide these two i get this ratio is b by a

so we see that area of any triangle inscribed on the ellipse is of constant ratio b by a to the area of the corresponding triangle p q r on the auxiliary circle therefore area of triangle abc is equal to b by a times area of triangle pqr now this p q r is a triangle inscribed in a circle of radius a

so we know when this has maximum area

so again recall this fact triangle p q r inscribed inside a circle has maximum area when the triangle is equilateral

so if i have this circle of radius a and then we want maximum possible area of this triangle p q r this happens exactly when this triangle is equilateral triangle

so if we have this p q r this is the center of the circle we are taking this radius to be equal to a and therefore we can find what is the maximum possible area of this equilateral triangle because if you look at this angle is pi by 6

so this is the side length will be 2 times a cos pi by 6 which is equal to 2 a into root 3 by 2 root 3 times a

so if a is the radius of this circle then the side length q r is root 3 times a and therefore maximum area of triangle p q r is equal to root three by four times side length square root three a square which is equal to this will be three root three by four a square and hence maximum area of triangle a b c will be b by a times this 3 root 3 by 4 a square which is equal to three root three by four times a b

so this is the maximum possible area of a triangle inscribed in an ellipse ok

so this brings us to the end of this lecture in the next lecture we will do some problems on hyperbola and rectangular hyperbola thank you you