

welcome to the lecture number eight on conic sections in the last lecture we studied about the equation of tangents and normal to a parabola in this lecture we will study about ellipse

so let me start with the points of intersection of line y equal to $m x$ plus c with the ellipse x square by a square plus y square by b square equal to one

so we have this ellipse this is a zero minus a zero this point is zero b and zero minus b and we have some line y equal to $m x$ plus c

so as you can see from the picture there are three possibilities this line might intersect the ellipse in two points or this line might be like this when it does not intersect the ellipse and the third case is that the line can intersect the ellipse in only one point

so we will derive the condition for all these three cases

so putting y equal to $m x$ plus c in the equation of ellipse gives x square by a square plus y is $m x$ plus c $m x$ plus c square by b square equal to one and then we have to solve for x from this equation

so this is same as b square x square plus a square m square x square plus $2 m c x$ plus c square equal to a square b square and this gives us a quadratic equation in x is x which is a square m square plus b square x square plus $2 a$ square $m c x$ plus a square c square minus b square equal to zero

so this is the this is a quadratic equation equation in x and its discriminant d is equal to b square minus $4 a c$ it is $2 a$ square $m c$ square minus 4 times a square m square plus b square times a square c square minus b square this gives discriminant is $4 a$ to the $4 m$ square c square minus $4 a$ to the $4 m$ square c square plus $4 a$ to the $4 m$ square b square minus $4 a$ square b square c square plus $4 a$ square b to the 4 this cancels and this can be written as 4 times a square b square is common and then we get a square m square minus c square plus b square

so we know that we have two points of intersections if d is strictly positive that is if c square is less than a square m square plus b square and there is only one point of intersection this means that the quadratic equation has real and equal root which is for d equal to zero that is c square equal to a square m square plus b square and no points of intersection if d is less than zero that is if c square is greater than a square m square plus b square

so one conclusion from this is the line y equal to $m x$ plus c is tangent to the ellipse x square by a square plus y square by b square equal to one if and only if c square is equal to a square m square plus b square

so this is the condition for the line to be tangent to the ellipse next let us try to find what is the length of the chord length of the chord joining two points lets say x one y one and x two y two on the ellipse

so suppose there are these two points on the ellipse this say x square by a square plus y square by b square equal to one then we want to find what is the length of the chord

so we have this ellipse and we have any two points we want this length

so what we have seen is that if we know the equation of the straight line the form y equal to $m x$ plus c then we know the points of intersections are given by roots of a quadratic equation

so the line the equation of the line joining x one y one and x two y two is given by y minus y one equal to slope is y two minus y one by x two minus x one times x minus x one

so this is same thing as y is equal to y two minus y one by x two minus x one times x plus y one minus y two minus y one by x two minus x one times x one

so that means that this is the value of m and this is c for this line now we know that x 1 and x 2 are roots of the equation x one and x two are the roots of the quadratic equation that we derived in the previous slide

so this is the equation whose roots are x_1 and x_2 .

so let us write it down $ax^2 + bx + c = 0$ now we know that if x_1 and x_2 are roots of a quadratic equation then we can write down formula for $x_1 + x_2$ and $x_1 x_2$

so this implies $x_1 + x_2 = -\frac{b}{a}$ and $x_1 x_2 = \frac{c}{a}$ where a, b, c are given as a verb now length of the chord is $l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ but this can be written as $x_1^2 + x_2^2 + y_1^2 + y_2^2$ and $y_1^2 = m^2 x_1^2 + c^2$ and $y_2^2 = m^2 x_2^2 + c^2$

so c cancels here and we get $l^2 = \sqrt{(x_1 - x_2)^2 + m^2(x_1^2 - x_2^2)^2}$

so we need to find what is $(x_1 - x_2)^2$ and then we know the length

so $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2 = \left(-\frac{b}{a}\right)^2 - 4\frac{c}{a}$

so if you substitute the value of $x_1 + x_2$ and $x_1 x_2$ this gives $l^2 = \frac{4a^2}{(a^2 + b^2)^2} [a^2 c^2 + b^2 (a^2 - 4ac + 4c^2)]$ this is $(x_1 - x_2)^2 + m^2(x_1^2 - x_2^2)^2$ is a square $\frac{4a^2}{(a^2 + b^2)^2} [a^2 c^2 + b^2 (a^2 - 4ac + 4c^2)]$ and on simplification this gives we have a square $(a^2 + b^2)^2$ in the denominator and the numerator becomes $4a^2 [a^2 c^2 + b^2 (a^2 - 4ac + 4c^2)]$ and therefore $l = \frac{2a}{a^2 + b^2} \sqrt{a^2 c^2 + b^2 (a^2 - 4ac + 4c^2)}$ is the positive square root of this

so that is equal to $2ab \sqrt{a^2 + b^2} \sqrt{a^2 c^2 + b^2 (a^2 - 4ac + 4c^2)}$ and length l is square root of one plus m^2 times this $(x_1 - x_2)^2$

so we can find the length of the chord joining any two points on the ellipse using this formula

so next we will derive what is equation of the tangent equation of the tangent at the point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$

so we will do this into in two different ways the first one we will do using the what we have just done

so so let the slope of the tangent at (x_1, y_1) be m then the equation of the line is $y - y_1 = m(x - x_1)$ that is $y = mx + c$ and we know the condition when this line is tangent

so let us put $c = y_1 - m x_1$

so we have equation of the line of the form $y = mx + c$ and we know the condition when this line is tangent

so we know that the line $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if and only if $c^2 = a^2 m^2 + b^2$ this is the condition for the line to intersect the ellipse in just one point in which case it will be tangent

so this is same thing as $(y_1 - m x_1)^2 = a^2 m^2 + b^2$ and this gives this you can see is a quadratic equation in m this is equal to $x_1^2 m^2 - 2x_1 y_1 m + y_1^2 - a^2 m^2 - b^2 = 0$ and this gives here note that the discriminant D is equal to $4x_1^2 y_1^2 - 4(x_1^2 - a^2)(y_1^2 - b^2)$ which gives $D = 4x_1^2 y_1^2 - 4x_1^2 y_1^2 + 4x_1^2 b^2 + 4a^2 y_1^2 - 4a^2 b^2$ which is same thing as $4a^2 y_1^2 + 4x_1^2 b^2 - 4a^2 b^2$ plus y_1^2 minus one but (x_1, y_1) lies on the ellipse

so this is equal to zero since x_1, y_1 lies on the ellipse
so thus the quadratic equation in m has say only one real root given by
therefore m is equal to $2x_1y_1$ divided by $2(x_1^2 - a^2)$ which is same as x_1y_1 divided by $x_1^2 - a^2$
so this is the value of the slope of the tangent line now we will write the equation

so the equation of the tangent line is $y - y_1 = m(x - x_1)$ is x_1y_1 by $x_1^2 - a^2$ times $x - x_1$ and this gives

so we have $y - y_1 = \frac{x_1y_1}{x_1^2 - a^2}(x - x_1)$ minus $x_1^2y_1$ by $x_1^2 - a^2$

so if we multiply this equation by y_1 we get $y_1y - y_1^2 = \frac{x_1y_1^2}{x_1^2 - a^2}(x - x_1)$ equal to $x_1y_1^2$ times y_1 by $x_1^2 - a^2$ minus $x_1^2y_1^2$ by $x_1^2 - a^2$ which i will write as $y_1y = \frac{x_1y_1^2}{x_1^2 - a^2}(x - x_1) + y_1^2$ equal to y_1^2 by $x_1^2 - a^2$ times $x - x_1$ and then i have plus y_1^2 times one minus $x_1^2y_1^2$ by $x_1^2 - a^2$

so let me rewrite it as $y_1y = \frac{y_1^2}{x_1^2 - a^2}(x - x_1) + y_1^2$ a square $x - x_1$ plus y_1^2 and this quantity is $x_1^2 - a^2$ minus x_1^2

so that will be minus a^2 by $x_1^2 - a^2$ minus a^2

so this is equal to y_1^2 by $x_1^2 - a^2$ times $x - x_1$ minus a^2 now note that $x_1^2 - a^2$ plus y_1^2 by b^2 square this is equal to one because it lies on the ellipse and this gives $b^2(x_1^2 - a^2) + y_1^2 = a^2$ square b^2 which means $b^2(x_1^2 - a^2) = a^2 - y_1^2$ equal to $a^2 - y_1^2$ square

so what we have here is y_1^2 by $x_1^2 - a^2$ minus a^2

so this will write as y_1^2 by $x_1^2 - a^2$ is equal to $\frac{a^2 - y_1^2}{a^2}$ minus of b^2 by a^2

so we put it back in this above equation to get $y_1y = \frac{a^2 - y_1^2}{a^2}(x - x_1) + y_1^2$ by a^2 minus a^2 which is equal to $\frac{a^2 - y_1^2}{a^2}(x - x_1) + y_1^2$ plus b^2 this we can rewrite as $x - x_1$ by a^2 by dividing the above equation by b^2 we get $x - x_1$ by a^2 plus y_1y by b^2 is equal to 1

so we will remember the equation of the tangent line at point x_1, y_1 to the ellipse as in this form

so this is equation of tangent at x_1, y_1

so next we will derive this equation again for those of you who have learnt calculus it can be derived much more simply

so another way using calculus

so we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ now if we differentiate this with respect to x we get $\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$ which implies $\frac{dy}{dx} = -\frac{b^2x}{ay}$ by a^2 times x by y now if you know that the slope of the tangent line is nothing but the derivative at the point x_1, y_1

so recall slope of tangent to a curve $y = f(x)$ at point x_1, y_1 is m equal to $\frac{dy}{dx}$ at the point x_1, y_1

so therefore the slope in this case will be $-\frac{b^2x_1}{ay_1}$ now once we know the slope we can write down the equation therefore equation of the tangent is $y - y_1 = m(x - x_1)$ is $-\frac{b^2x_1}{ay_1}(x - x_1)$ which gives $y - y_1 = -\frac{b^2x_1}{ay_1}(x - x_1)$ multiply by y_1 this gives $y_1y - y_1^2 = -\frac{b^2x_1y_1}{a}(x - x_1)$ equal to $-\frac{b^2x_1y_1}{a}(x - x_1)$ plus $b^2x_1^2$ by a^2 minus $b^2x_1^2$ by a^2

so this gives $y_1y + \frac{b^2x_1^2}{a^2} = \frac{b^2x_1y_1}{a}(x - x_1) + y_1^2$

plus b^2 by a^2 times x^2 plus y^2 which is same thing as writing y^2 by b^2 plus x^2 by a^2 equal to y^2 by b^2 plus x^2 by a^2 but this is equal to one because (x, y) lies on the ellipse

so we get the same equation $x^2/a^2 + y^2/b^2 = 1$

so this is the equation of the tangent line at the point (x_1, y_1) to the ellipse $x^2/a^2 + y^2/b^2 = 1$ next we will derive the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point (x_1, y_1)

so what is normal to a curve it is the line which is perpendicular to the tangent line

so this is a tangent line and this one is normal

so we already know that the slope of the tangent we saw that the slope of tangent at (x_1, y_1) is $-b^2 x_1 / a^2 y_1$ therefore slope of the normal at (x_1, y_1) is given by $m = a^2 y_1 / b^2 x_1$ which is a square y_1 by b^2 times x_1 once we know the slope we can write down the equation therefore the equation is $y - y_1 = m(x - x_1)$ which we will write in this form that is $x - x_1$ divided by x_1 by a^2 equal to $y - y_1$ by y_1 by b^2 this is the equation of the normal line next we will discuss the parametric form for any general point on the ellipse parametric form of a general point on the ellipse $x^2/a^2 + y^2/b^2 = 1$

so any point (x, y) on the ellipse satisfies $x^2/a^2 + y^2/b^2 = 1$

so from this we can easily see that we need the x by a and y by b such that the sum of the square is equal to one we know that $\cos^2 \theta + \sin^2 \theta = 1$

so putting $x/a = \cos \theta$ and $y/b = \sin \theta$ we see that $x = a \cos \theta$ $y = b \sin \theta$ gives any point on the ellipse for θ we can take between zero and two π

so the parametric form will take as the any general point p on the ellipse can be written as $(a \cos \theta, b \sin \theta)$ now let us see this little more carefully

so what is this angle θ for any given point how do we

so let's define here

so we will call we will call the angle θ to be the eccentric angle of the point p

so let us draw the ellipse suppose this is our ellipse $x^2/a^2 + y^2/b^2 = 1$ then we have this point is $(a \cos \theta, b \sin \theta)$ this is the origin O now here is a general point p whose coordinates are $(a \cos \theta, b \sin \theta)$ and where do we see this angle θ in the picture

so now if you see this the x coordinate of the point is $a \cos \theta$

so this length is $a \cos \theta$ now if i want to take this point

so if we take angle θ with the positive x axis then the x and y coordinates are given by let me write this point q as $(a \cos \theta, a \sin \theta)$

so if i take this angle θ then this point q is has the coordinate $(a \cos \theta, a \sin \theta)$ in this case we are taking b to $b < a$

so this point is outside this ellipse now where is this point q

so if you see the this point q lies on this circle

so the point q lies on the circle $x^2 + y^2 = a^2$

so this is the circle which is again centered at the origin but the radius is equal to one

so if i draw or this circle this is the circle which has radius equal to a and center is same as centered of the center of the ellipse

so this point q lies on the circle and has the same x coordinate as the point p this is the circle whose radius is a

so we can find this angle let theta by

so any point on the ellipse x^2 by a^2 plus y^2 by b^2 equal to one this can be obtained by taking the point on the circle x^2 plus y^2 equal to a^2 which has the angle theta with the positive x axis and then we drop the perpendicular to the x axis and then this intersects at the point p on the ellipse whose coordinates are $a \cos \theta$ $b \sin \theta$ any point on the ellipse given by $a \cos \theta$ $b \sin \theta$ is on the vertical line through the point on the circle x^2 plus y^2 equal to a^2 at an angle theta with the positive x axis and what is the ratio of the y coordinates of this if you see is $\sin \theta$ and $b \sin \theta$ are the y coordinates of q and p respectively

so the ratio of the y coordinates of p and q is $b \sin \theta$ divided by $a \sin \theta$ which is equal to b by a

so in fact you can use this to describe the ellipse

so if i want to describe the ellipse x^2 by a^2 plus y^2 by b^2 equal to one you first look at the circle of radius a centered at origin and then you look at any point q on this circle then the point p on the ellipse will be on this vertical line through this q and where is this point p this is such that the ratio

so if i call this q m then the the ratio p m divided by q m is equal to b by a so that means that p m is equal to b by a times q m

so you take this point p with whose y coordinate is b by a times the y coordinate of q and if you keep varying the point q on the circle what you get is the ellipse x^2 by a^2 plus y^2 by b^2 equal to one

so next we will do one problem determine the locus of the point of intersection of tangents to the ellipse x^2 by a^2 plus y^2 by b^2 equal to one which meet at right angles

so what we we want is to determine the locus of the point of intersections of tangent with meet at right angle

so let let us write what is the equation of the tangent

so solution equation of the tangent whose slope is m is given by y equal to $m x$ plus c with c^2 equal to $a^2 - m^2 b^2$ that is we have equation is y equal to $m x$ plus square root of $a^2 - m^2 b^2$

so once you know the slope you can write the equation of the tangent similarly what is the equation of the tangent to slope is minus one by m

so the equation of the tangent which is perpendicular to to the above tangent is y equal to

so the slopes would be minus one by m

so minus one by m times x plus a square times minus one by m square plus b square let's write this equation one and this is our equation two y equal to minus one by m x plus a square by m square plus b square square root

so if h comma k is a point of intersection of these two tangent if h comma k is the point of intersection of one and two then we have $k - m h$ is equal to square root $a^2 - m^2 b^2$ and $m k + h$ is equal to square root $a^2 - m^2 b^2$

so h k lies on both the equation one and two this gives us these two equations now to find the the locus of the point of intersection we need to eliminate m from the above two equations let us call this three and four

so we need to eliminate m from equations three and four

so if you squaring square and add three and four squaring and adding equation three and four gives $k^2 - m^2 h^2 + mk + h^2 = a^2 + m^2 + b^2 + a^2 + b^2 + m^2$ and this gives $k^2 + h^2 = a^2 + b^2 + m^2$ on simplification and that is same thing as $x^2 + y^2 = a^2 + b^2$ therefore the locus is $x^2 + y^2 = a^2 + b^2$

so locus is a circle this is a circle centered at zero zero and radius square root of $a^2 + b^2$ ok

so we will stop here for this lecture in the next lecture we will discuss some more problems on ellipse and then we will talk about tangents and normal to hyperbola thank you