

welcome to lecture 7 on conic sections

so in the last lecture we discussed about tangent and normal of a parabola in this lecture we will describe the parametric form of parabola and then we will discuss the tangent normal in terms of the parametric form parametric form of the parabola $y^2 = 4ax$

so note that this parabola x is always non negative and y can be negative as well as positive

so and as x you increase then you get for every fixed x there are two values of y here and as x goes from 0 to infinity y takes values from if you look at the upper half part of the parabola y goes again from 0 to infinity in the lower half it goes from zero to negative infinity right

so if we put y equal to t where t is any real number

so y can take value from minus infinity to positive infinity then x is equal to y^2 by $4a$

so this is equal to t^2 by $4a$

so any general form

so any general point

so any general point on the parabola can be written as x is t^2 by $4a$ and y is t now instead of writing like this this involves fraction t^2 by $4a$ suppose we do not want this fraction then we can put

so instead of y equal to t if we use y equal to $2at$ note that a is positive

so this again is t goes from minus infinity to infinity y is also from minus infinity to infinity

so then x will be equal to y^2 is two eight t^2 by four a that is $4a$ square t^2 by four a which is a times t^2

so we will write we will use this as the parametric form we will use x equal to at^2 y equal to $2at$ as the parametric form of the parabola $y^2 = 4ax$ that is any general point of the parabola $y^2 = 4ax$ can be written as at^2 $2at$

so the advantage of this is that we use only one parameter t to describe any point on the parabola this already satisfies if you write a general point as x_1 y_1 then you have to use one more equation that $y_1^2 = 4ax_1$

so this is the general point and you can see that this at t equal to zero you get the vertex zero zero this is at t equal to zero if t is any positive then you get x equal to at^2 y equal to $2at$ that means y is positive

so this is point at^2 $2at$ for t positive and we have point at^2 $-2at$ for t negative

so as you take t varying from minus infinity to infinity you get a point on the parabola and for every t you get only one point

so as t varies between minus infinity to plus infinity we get all points on the parabola also there is a unique point at^2 $2at$ on the parabola for any given t

so all the points can be uniquely represented as at^2 $2at$ now let us write the equation of the tangent and normal in terms of this parametric form equation of tangent and normal in parametric form recall equation of tangent at point x_1 y_1 is given by $yy_1 = 2a(x + x_1)$ and equation of normal at x_1 y_1 is given by $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ you can see easily from this equation is $2a$ by y_1

so slope of normal is minus y_1 by $2a$ times $x - x_1$

so these are two equations of tangent and normal using x_1 y_1 equal to at^2 $2at$ we get equation of tangent is y times y_1 is two eight equal to $2a(x + x_1)$ $2at^2$ $2at$ cancels this gives that is t times y equal to $x + at^2$

this triangle

so we need to prove that this angle and this angle are equal

so what we do is let's say let me use another color

so let us draw perpendicular from t to this line spn sq let's call this point as m and on t cube we are perpendicular m dash ok

so we know that we can take the point in the parametric form

so let p be $a t_1^2$ $2 a t_1$ and q be the point $a t_2^2$ $2 a t_2$.

then we just derived the intersection of the tangent line

so by the previous formula for the point of intersection of tangent lines at p and q we have the coordinate of point t is given by $a t_1 t_2$ and $a t_1 + t_2$

so we have got the coordinate of point t now let us write what is the equation of the line sp we have this figure

so line joining the focus to this point p what is the equation of this line is $y - 0 = \text{slope} \frac{y - 0}{x - a}$ because s is the point $a, 0$ and p is the point $a t_1^2, 2 a t_1$

so this gives $y = \frac{2 t_1}{t_1^2 - 1} (x - a)$ that is $t_1^2 y = 2 t_1 x - 2 a t_1$

now let m be the foot of perpendicular from t to the line to the line joining s and p then what is the length of this perpendicular recall the perpendicular distance of a point x_1, y_1 to the line $a x + b y + c = 0$ is given by $\frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$

so therefore we can calculate the distance tm this is equal to the equation of the line is given by this previous equation

so we put x_1, y_1 as $a t_1 t_2, a t_1 + t_2$ in this equation this is equal to $\frac{|a t_1 t_2 - 2 t_1 (a t_1 + t_2)|}{\sqrt{t_1^2 - 1}}$

so we put $y_1 = a t_1 t_2 + t_2^2$ this is $t_1^2 y_1 - 2 t_1 x_1 = a t_1 t_2^2 + t_2^3 - 2 t_1^2 x_1$ this in absolute value divided by square root of $t_1^2 - 1$

so if you simplify this this is equal to a times a we can take common from all this and the denominator if you see this is $t_1^2 - 1$ whole square this will be sorry this was $t_1^2 - 1$ square plus $2 t_1$ whole square this is $t_1^2 + 1$ whole square under square root and the numerator is a times $|t_1^2 t_2^2 - 2 t_1^2 t_2 - t_2^3|$ let's simplify it further a times we multiply this out

so this is $\frac{a |t_1^2 t_2^2 - 2 t_1^2 t_2 - t_2^3|}{\sqrt{t_1^2 + 1}}$ and then plus $t_1^2 - 1$ now you can factor and this is equal to $a |t_1^2 t_2^2 - 2 t_1^2 t_2 - t_2^3|$ times $\frac{1}{\sqrt{t_1^2 + 1}}$ this cancels and you get tm is equal to $a |t_1^2 t_2^2 - 2 t_1^2 t_2 - t_2^3|$

so this is the length of the perpendicular drop from the point of intersection of the tangent to the line joining the focus to the point p since this is symmetric symmetric with respect to t_1 and t_2 the distance set tm dash where m dash is the foot of perpendicular to the line sq this is also equal to tm equal to $a |t_1^2 t_2^2 - 2 t_1^2 t_2 - t_2^3|$

so tm is same as tm dash now let us look at the picture again you have a point q here is a point p the tangents meet at this point t here and we have focus s minus p nsq this point is m and this point is m dash

so what we have got is tm is equal to tm dash we have tm equal to tm dash

x_2, y_2 and x_3, y_3 if we drop these perpendiculars these are points p, q and r and let us call this as a, b and c area of the triangle pqr can be calculated by area of trapezium pab plus area of the trapezium rbq minus area of trapezium pbc

so this area we wanted to calculate

so for that we find the area of the two trapezium and then subtract this area so area of the trapezium can be calculated easily here this is equal to half times here these two are opposite sides of the trapezium of length y_1 and y_3

so half $y_1 + y_3$ times this distance here is $x_3 - x_1$ $x_3 - x_1$ plus half of this one will be $y_2 + y_3$ times this distance is $y_2 - y_1$ minus x_3 .

minus the area of this trapezium is half $y_1 + y_2$ times this is $x_2 - x_1$ and on simplification we get this area of the triangle is equal to half times $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$ easy way to remember this is this is equal to half of determinant of $x_1, x_2, x_3, y_1, y_2, y_3$ and the third column is one one one

so thus area of triangle pqr is equal to half times $x_1^2 y_2 - y_3$

so $2x_1^2 y_2 - 2x_1^2 y_3 + x_2^2 y_3 - y_1^2 x_2 + x_3^2 y_1 - y_2^2 x_3$ which is equal to $2x_1^2$ if you take out 2 cancels with half

so $x_1^2 y_2 - y_3 + x_2^2 y_3 - y_1^2 x_2 + x_3^2 y_1 - y_2^2 x_3$ which is equal to $x_1^2 y_2 - y_3 + x_2^2 y_3 - y_1^2 x_2 + x_3^2 y_1 - y_2^2 x_3$ this is the area of the triangle formed by any three points on the parabola similarly we have to find the area of the triangle formed by the tangents at these points

so we know that point of intersection of tangent points of intersection of tangents are given by $(t_1 + t_2, t_1 t_2)$ if you take second and third point then you get $(t_2 + t_3, t_2 t_3)$ and first and third point will give $(t_3 + t_1, t_3 t_1)$ and using the formula for the area we can see that

so area of the triangle formed by these tangents is equal to half of $(t_1 + t_2)(t_2 + t_3) - (t_1 + t_3)(t_2 t_3)$ the difference in the y coordinate will be $t_3 - t_2$ plus $t_3 t_1$ times $t_1 - t_2$ which if you simplify this is equal to half of $(t_1 - t_2)^2 (t_1 + t_2 + t_3)$ and $t_1 - t_3$ which is equal to half of area of pqr

so this finishes today's lecture in the next lecture we will discuss about tangents and normal or ellipse and hyperbola etcetera thank you