

welcome to lecture 6 on conic sections

so first we will find out the points of intersection of the line y equal to $m x$ plus c with the parabola y square equal to $4 a x$

so if we see this parabola y square equal to $4 a x$ is the parabola vertex is at the origin and the axis is the x axis

so this is the equation of the parabola y square equal to $4 a x$ and suppose we have some line y equal to $m x$ plus c then there are three possibilities we can have this line intersecting the parabola in two points or we can have the line which intersects the parabola in only one point in which case this line is tangent to the parabola or it can happen that the line does not intersect the parabola for example this

so we have three cases this is line 1 one 1 two 1 three

so three possibilities which we can see from the graph are first two points of intersection second is only one point of intersection and third is no points of intersection

so we would like to deduce the condition for all these three cases algebraically

so to find the point of intersections points of intersection of y equal to $m x$ plus c and parabola y square equal to $4 a x$ we solve we put y equal to $m x$ plus c in the equation of the parabola then we get $m x$ plus c squared equal to $4 a x$ this is equivalent to writing m square x square plus $2 m c x$ plus c square equal to $4 a x$ that is m square x square plus two times $m c$ minus two $a x$ plus c squared equal to 0 .

so the roots of x of this quadratic equation will give the x coordinate of the point of intersection and then we can find the y coordinate using the equation y equal to $m x$ plus c now this is a quadratic equation

so this is a quadratic equation in x

so it has either two real and distinct roots or equal roots which means there is only one real root or two non real complex roots the condition for that will be if we find the discriminant discriminant of this quadratic equation is given by $2 m c$ minus $2 a$ square minus 4 times m square c square which is equal to 4 you can take out and then we have m square c square minus $4 a m c$ plus $4 a$ square minus m square c square

so m square c square can be cancelled and this is equal to 16 times a square minus $a m c$

so therefore there are two points of intersections if a square minus $a m c$ is greater than 0 that is a is greater than $m c$

so this is the condition when discriminant is positive we have two points of intersections there is one point of intersection if the discriminant is equal to 0 that means a equal to $m c$ and no points of intersection if a is less than $m c$

so these are the conditions if a is greater than $m c$ then we get two values of x

so we get two points of intersection if a is equal to $m c$ then we get only one point of intersection and the line is tangent to the parabola and if a is less than $m c$ then there is no point of intersection

so when a is equal to $m c$ the line y equal to $m x$ plus c intersects the parabola in only one point and hence it is tangent to the parabola y square equal to $4 a x$ what is the point at which this is tangent at the point we can see from this quadratic equation that when the discriminant is equal to 0 then x can be solved the point x comma y given by x is equal to for quadratic equation $a x$ square plus $b x$ plus c equal to zero for equal roots we get minus b by two a

so that is minus two $m c$ minus two a divided by two m square but in this case a is equal to $m c$

so we can write this as $\frac{-2mc - 2mc}{2m^2}$
so we get this is equal to $\frac{c}{m}$ and putting $x = \frac{c}{m}$ in the equation $y = mx + c$ gives $y = m \times \frac{c}{m} + c$ which is equal to $2c$

so the line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$ at the point $(\frac{c}{m}, 2c)$ provided $mc = a$
so the equation of the line can be written as the equation of the tangent line is $mx + c = a$ and m and c are related by this equation $mc = a$ next let us find the length of the chord of a parabola $y^2 = 4ax$

so if we have this parabola $y^2 = 4ax$ and if we take any two points on this parabola what is the length of this chord pq

so suppose p has coordinates (x_1, y_1) and q has coordinates (x_2, y_2) and suppose the equation of this line is $y = mx + c$

so length of the chord joining (x_1, y_1) and (x_2, y_2) on the parabola $y^2 = 4ax$ is $l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ now what we know is that the equation of the line joining (x_1, y_1) and (x_2, y_2) is $y - y_1 = m(x - x_1)$

so let me write in the form $y - y_1 = m(x - x_1)$

so this can be written as $y = mx + c$ where $m = \frac{y_1 - y_2}{x_1 - x_2}$ and $c = y_1 - mx_1$ this we are taking if $x_1 \neq x_2$ that means that if we take this chord such that the x coordinate of the p and q line is different then because otherwise if $x_1 = x_2$ then it is just the difference in the y coordinate which is the length of the curve

so that can easily be calculated

so note that if $x_1 = x_2$ then length of chord l is equal to $|y_1 - y_2|$ and since we have $y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$ and $x_1 = x_2$ then $y_1 = -y_2$ and $y_1 - y_2 = 2y_1$ in absolute value is just $2\sqrt{4ax_1} = 4\sqrt{ax_1}$ but in the case where $x_1 \neq x_2$ then this line is not parallel to the y axis and we have this line now what we have seen is that x satisfies

so x_1 and x_2 satisfy the quadratic equation that we have derived is $m^2x^2 + 2mcx - a^2 = 0$ therefore $x_1 + x_2 = \frac{-2mc}{m^2}$ the sum of the roots of the quadratic equation is given by $-\frac{b}{a}$

so $x_1 + x_2 = \frac{-2mc}{m^2}$ and $x_1 x_2 = \frac{-a^2}{m^2}$ the product of the roots is $\frac{c^2}{m^2}$

so $x_1 - x_2 = \frac{4a^2 - 4mc^2}{m^2}$

so therefore $(x_1 - x_2)^2 = \frac{16a^2 - 16mc^2}{m^4}$ which is equal to $\frac{16}{m^4} (a^2 - mc^2)$ and $(y_1 - y_2)^2 = 4m^2(x_1 - x_2)^2 = \frac{64}{m^2} (a^2 - mc^2)$

so this can be written as $l^2 = \frac{64}{m^2} (a^2 - mc^2)$ here we are taking m to the 4 out

so $l^2 = \frac{16}{m^2} (a^2 - mc^2)$

so this cancels and we get this is equal to $16a^2 - 16mc^2$ divided by m^2 and $y_1 - y_2 = m(x_1 - x_2) = \frac{4}{m} (a^2 - mc^2)$

so this is equal to $4\sqrt{a^2 - mc^2}$

so therefore the length of the chord in terms of m and c is given by $l = \frac{4}{m} \sqrt{a^2 - mc^2}$ which is equal to

square root of $x_1^2 - x_2^2 + m^2 x_1^2 - x_2^2$ which is equal to square root of $1 + m^2$ times $|x_1 - x_2|$ and we have got $|x_1 - x_2|$ is sixteen a times a minus $m^2 c$ by m^2 to the four

so this is equal to square root of $1 + m^2$ times four by m^2 square root of a times a minus $m^2 c$ ok

so this gives the length of the chord in terms of x_1 and x_2 we can put the value of m and c in terms of x_1, y_1 and x_2, y_2 to get the formula for l in terms of x_1, y_1 next to y_2 we can put m equal to $\frac{y_1 - y_2}{x_1 - x_2}$ and c is equal to $\frac{y_1^2 - y_2^2}{x_1 - x_2}$ ok next we will find the equation of the tangent line

so equation of the tangent line to the parabola $y^2 = 4ax$ at a point (x_1, y_1)

so let us take a parabola and suppose (x_1, y_1) is some point on the parabola we want to find the equation of this tangent line at this point (x_1, y_1)

so suppose the line has slope m then the equation of the line

so now the tangent line is given by $y - y_1 = m(x - x_1)$ that is $y = mx + y_1 - mx_1$

let us put this equal to c we know that this line is tangent to $y^2 = 4ax$ if $a = mc$ that is $a = m(y_1 - mx_1)$ this is equivalent to writing $m^2 x_1 - m y_1 + a = 0$

so what is the value of m from this this gives $m = \frac{y_1 \pm \sqrt{y_1^2 - 4ax_1}}{2x_1}$ but $y_1^2 = 4ax_1$ because (x_1, y_1) lies on the parabola

so $m = \frac{y_1}{2x_1}$

so the slope of this tangent line is $m = \frac{y_1}{2x_1}$ this can also be written as $m = \frac{y_1^2}{2x_1 y_1}$ but $y_1^2 = 4ax_1$ so $m = \frac{2ax_1}{2x_1 y_1} = \frac{a}{y_1}$

so x_1 cancels here and we get $2a$ divided by y_1

so we get this formula $m = \frac{2a}{y_1}$ and the equation of the tangent line is $y = mx + c$ was $y_1 - mx_1$ which is equal to m is given by $\frac{2a}{y_1} x + y_1 - m x_1$ if you see from this first equation $m x_1 = \frac{y_1}{2}$

so this is equal to $2ax + \frac{y_1^2}{2}$ this implies if we multiply by y_1 $y_1^2 = 2ax y_1 + \frac{y_1^3}{2}$ now let us put $y_1^2 = 4ax_1$

so this is $2ax + \frac{y_1^2}{2}$ that is $y_1^2 = 2ax + \frac{y_1^2}{2}$

so this is the equation of the tangent line at the point (x_1, y_1) this is equation of the tangent line to the parabola $y^2 = 4ax$ at the point (x_1, y_1) on the parabola ok

so we have derived the equation of the tangent line to the parabola $y^2 = 4ax$ at any point (x_1, y_1) on the parabola this can also be derived using calculus if you have learnt about derivatives

so let us derive this derivation of the equation of the tangent line using calculus we have the equation $y^2 = 4ax$ this is the equation of the parabola also (x_1, y_1) lies on this (x_1, y_1) lies on the parabola this implies $y_1^2 = 4ax_1$ now we know that the slope of the tangent line at a point (x_1, y_1) on any curve $y = f(x)$ is given by the slope is equal to the derivative $\frac{dy}{dx}$ at the point (x_1, y_1)

so all you have to do is calculate the derivative $\frac{dy}{dx}$

so $y^2 = 4ax$ if we differentiate this with respect to x we get $2y \frac{dy}{dx} = 4a$ which implies $\frac{dy}{dx} = \frac{2a}{y}$

which will imply m is equal to the derivative $\frac{dy}{dx}$ at the point x_1, y_1
 so this is equal to $2a/y_1$ note that we got this same formula m equal to $2a/y_1$ using our previous method but that involved more algebra whereas if you know calculus you can get this easily and therefore the equation therefore equation of the tangent line is $y - y_1 = m(x - x_1)$ which implies $y - y_1 = 2a/y_1(x - x_1)$ which implies $y - y_1 = 2a/x_1 - 2a/x_1$ but $y_1^2 = 4ax_1$ therefore $y - y_1 = 2a/x_1 - 2a/x_1$ which is same as $y - y_1 = 2a/x_1 - 2a/x_1$ or $y - y_1 = 2a/x_1 - 2a/x_1$

so remark if if you note this derivation the above derivation assume that this point x_1, y_1 is different from the origin $0, 0$ because we are writing $2a/y_1$ or $2a/x_1$

so so this assume this but however if the point x_1, y_1 is $0, 0$
 so this is the vertex of the parabola $y^2 = 4ax$ and if we take this point $0, 0$ then it is clear that the tangent line at $0, 0$ is the y axis because y axis intersects this parabola at only one point $0, 0$

so this equation is the y axis whose equation is $x = 0$
 so if we put $x_1, y_1 = 0, 0$ in the equation obtained which was $y - y_1 = 2a/x_1(x - x_1)$ we get left hand side is $0 = 2a/x_1(x - 0)$ that is $0 = 2ax$ hence the equation $y^2 = 4ax$ is valid for even the point $0, 0$

so this is the general equation of tangent at point x_1, y_1 next we will find the equation of the normal line at any point x_1, y_1 equation of the normal at point x_1, y_1 on the parabola $y^2 = 4ax$

so what is the normal line
 so if we have this parabola if we take a point x_1, y_1 the tangent line is this and the normal line is the line which is perpendicular to the tangent line
 so this line we have this is the normal line and this is the tangent line the equation of the tangent is $y - y_1 = 2a/x_1(x - x_1)$

so if we assume if y_1 is non zero then we can write this as $y = 2a/x_1 + y_1$
 so therefore the slope of tangent line is given by $2a/y_1$ which implies slope of the normal line let us call it m this is equal to $-y_1/2a$ because these two lines are perpendicular the product of the slope is equal to -1

so we get slope is $-y_1/2a$ therefore equation of normal is $y - y_1 = -y_1/2a(x - x_1)$ let us write this equation in terms of the slope m

so in terms of the slope m we have $m = -y_1/2a$ that is $y_1 = -2am$ and therefore $x_1 = y_1^2/4a$ which is equal to $4a^2m^2/4a$ or this is equal to am^2

so putting in the equation for normal which is $y - y_1 = m(x - x_1)$ we get $y - y_1 = m(x - x_1)$
 so $y + 2am = m(x - am)$

so $x - am = (y + 2am)/m$ which is same thing as writing $y = m(x - am) + 2am$

so this is equation of the normal at the point x_1, y_1 which in terms of m $x_1 = am^2$ and $y_1 = -2am$

so if we write a general point as $am^2, -2am$ on the a parabola then the equation can be written as $y = m(x - am) + 2am$ ok

so next we will define what is called sub tangent and sub normal and find the length for the parabola sub tangent and sub normal we have parabola $y^2 = 4ax$

equal to four a x suppose there is a point p on the parabola let us look at the tangent line at this point p and suppose this tangent line intersects the x axis at a point t and also let us draw the normal line and suppose the normal line intersects the x axis in n this is perpendicular then this p t ok then we also draw perpendicular from this p to the x axis let us call that point as a

so p t is the tangent between the point p and the interest section point on the x axis and the projection of this on the x axis is eighty

so this a t will be called sub tangent and a n is this sub normal this again if you look at this p n is the normal and its projection on the x axis is in

so this is sub tangent and this one a n is sub normal

so if we take the coordinates of point p as x one y one we know the equation of the tangent line line p t is $y - y_1 = m(x - x_1)$ $y - 1 = 2(x - 1)$ $y = 2x - 1$

so the coordinates of t can be put obtained by putting y equal to zero putting y equal to zero gives x equal to minus x one

so therefore t is the point minus x one comma zero this point is minus x one zero note that this point a is x one comma zero because p is x one y one

so this point t is equidistant from the vertex o and this point a this vertex is equidistant from t and a

so tup sub tangent a t is equal to two times x one and what is a n this sub normal

so to find the this sub normal note that this triangle t a p is similar to triangle p a n why is this because if you see suppose this angle we call theta and this is say p a t is a right angle triangle

so this angle will be pi by 2 minus theta again this is 90 degree

so this angle is theta

so angle le a t p is same as angle a p n and then there is one ninety degree is common

so therefore these two triangles are similar and therefore if i write a n divided by a p this is same thing as a p divided by a t in the similar triangles we know that the ratios are the same

so this implies a n equal to a p square by a t but what is the length a p a p is nothing but y one this is y one square by a t is two x one but y 1 square is 4 a x 1 divided by 2 x 1

so this is constant 2 a

so therefore sub normal a n is equal to two a which is a constant

so sub normal does not depend on the point x one y one whereas sub tangent depends on the x coordinate of the point x one y one

so we will stop here for this lecture in the next lecture we will do some more problems involving tangent normal etcetera thank you you