

hello everyone this is the fifth lecture on conic sections

so we will talk about parabola ellipse and hyperbola

so let me first talk about the way we know that the standard forms of parabola we saw is of the form  $y^2 = 4ax$  or  $y^2 = -4ax$  or  $x^2 = 4ay$  or  $x^2 = -4ay$

so these are given by this is the parabola  $y^2 = 4ax$  this has vertex at the origin all this parabola has vertex at the origin and then we have this is the parabola  $x^2 = 4ay$  and then we have this parabola if i reflect in the y axis i get  $y^2 = -4ax$  and this is  $x^2 = -4ay$  let us look at a parabola of this is type  $x^2 = 4ay$  or  $x^2 = -4ay$

so here you see that y is given by  $x^2 = 4ay$  or  $x^2 = -4ay$  right

so this is a special type of quadratic equation in x y is given as a quadratic in x

so this gives  $y = \frac{1}{4a}x^2$  or  $x^2 = 4ay$  or  $y = \frac{1}{4a}x^2$  now if we consider general quadratic polynomial in x

so consider the general quadratic  $y = ax^2 + bx + c$  where a is given to be non zero now we will show that this again represents a parabola not in the standard form the vertices need not be at the origin but how to find the vertices of this parabola

so we can write this as  $y = a(x^2 + \frac{b}{a}x + \frac{c}{a})$  and then what we do is we complete a square here

so you can write this as  $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$

so this will give you  $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$

so we subtract  $\frac{b^2}{4a^2}$  plus  $\frac{c}{a}$  which is same thing as a times  $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}$  multiplied by a gives  $b^2/4a + c/a$

so this is  $y = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + \frac{c}{a}$  or this can be written as  $y + \frac{b^2}{4a} - \frac{c}{a} = a(x + \frac{b}{2a})^2$

so this is of the form  $y - k = a(x - h)^2$  where k is negative of this

so  $4ac - b^2/4a$  and h is  $-b/2a$

so this is of the form  $y - k = a(x - h)^2$  if i put  $y - k$  as y' then we have  $y' = a(x - h)^2$

so if we draw the graph of this parabola you see the vertex is shifted to h comma k

so if my suppose a is positive then this parabola will be opening up and the vertex is at point h comma k which can be anywhere here we are taking in the fourth quadrant and then the parabola will be like this right

so this this is the vertex and here the axis of the parabola will be this line  $x = h$  the parabola is symmetric about this line and the focus will lie a on this axis and in this case the directrix will be a line  $y = k - l$

so its not difficult to find the focus and directrix also we know that the parabola is given by all set of all points whose distance from the focus and directrix is same

so since we have this vertex here is h comma k this distance from the vertex to the directrix this is  $l - k$  and the distance of focus to this if my focus is say at the point h comma alpha then the distance from focus to vertex this is  $alpha - k$

so because the vertex is equidistance from focus and the directrix we have  $alpha - k = l - k$  which means  $alpha = l$  this should be mod of  $l - k$

so this this is  $k - 1$

so  $\alpha + 1$  is equal to  $2k$  this is one equation we get and if you take any other point on the parabola let us say we take the point  $p$  which is some  $x$  comma  $y$  is  $\alpha$  then we have the equation of parabola is  $y - k$  is  $a$  times  $x$  minus  $h$  square

so putting this  $x$  comma  $\alpha$  here we have  $\alpha - k$  is equal to  $a$  times  $x$  minus  $h$  square which will give  $x - h$  square is equal to  $\alpha - k$  by  $a$  or  $x$  is equal to  $h$  plus minus square root of  $\alpha - k$  by  $a$

so  $p$  point is if we take this point  $p$  this has coordinate  $h$  plus square root of  $\alpha - k$  by  $a$  comma  $\alpha$  now what is the the distance of  $p$  to  $f$  distance of  $p$  to the focus is equal to this difference in the  $x$  coordinate

so this will be square root of  $\alpha - k$  by  $a$  and what is the distance of  $p$  to the directrix  $p$  to the line  $l$  this distance is the  $y$  coordinate here is  $\alpha$  and this is  $y$  equal to  $1$

so  $\alpha - 1$  because we are taking here this  $y$  equal to  $1$

so  $\alpha - 1$  now  $p f$  equal to  $p l$  will imply  $\alpha - k$  by  $a$  is equal to  $\alpha - 1$  square and from these two equation we can solve for  $\alpha$  and  $1$

so let us put this is equation one one implies  $k$  is equal to sorry ah let us put from 1 let us write  $1$  in terms of  $\alpha$   $2k - \alpha$  and then putting this in this equation  $2\alpha - k$  by  $a$  equal to  $\alpha - 2k - \alpha$  square which means  $\alpha - k$  is  $a$  times this is  $2\alpha - 2k$

so  $4\alpha - k$  squared this implies  $\alpha - k$  is equal to one by four  $a$  since  $\alpha$  is different from  $k$

so you see  $h$  comma  $a$  is here the focus is different from the vertex

so this is since  $\alpha - k$  is non-zero

so we cancel  $\alpha - k$  i get  $\alpha - k$  equal to one by four  $a$  which implies  $\alpha$  is equal to  $k$  plus one by four  $a$  and if you use this this will give  $1$  is equal to  $2k - \alpha$  which is  $k - 1$  by four  $a$

so for the parabola  $y - k$  equal to  $a$  times  $x - h$  squared the vertex is at  $h$  comma  $k$  focus is that focus was at  $h$  comma  $\alpha$  which is equal to  $h$  comma  $\alpha$  we got as  $k$  plus one by four  $a$  and directrix is the line  $y$  equal to  $1$  that is  $y$  equal to  $k - 1$  by four  $a$

so to remember this if we have this vertex is  $h$  comma  $k$  that means that this distance is minus  $k$

so this directrix is at distance  $1$  by four  $a$  from this vertex and the focus is again at distance one by four  $a$  you know that the vertex is the midpoint between this focus and the this point on the directrix

so the distance between focus and the directrix in this case is two  $a$  one by two  $a$  ok

so we we have seen that we have defined parabola by using focus and directrix

so parabola is set of all points such that the distance of any point to the focus is equal to the distance of the point perpendicular distance of the point from the directrix whereas we have defined the ellipse and hyperbola in terms of for two fixed points called the foci and then we defined

so let let us look at ellipse and hyperbola in terms of focus and directrix

so let  $f$  be any fixed point let us take on the  $x$  axis

so  $f$  is any  $f$  comma zero and suppose we take let us take ellipse

so we have  $f$  comma zero or we we write this as  $c$  comma zero and minus  $c$  comma zero and then ellipse with major axis on the  $x$  axis is given by this this is a comma zero minus  $a$  zero this is zero  $b$  zero minus  $b$

so if we look at this ellipse  $x$  square by  $a$  square plus  $y$  square by  $b$  square equal to one where  $a$  is greater than  $b$  and  $c$  square is  $a$  square minus  $b$  square in this case now let us look at this line

so consider the line  $x$  equal to  $a$  square by  $c$

so where is this line this is some line  $x$  equal to  $a^2$  by  $c$  note that since  $a$  is greater than  $c$  this will be greater than  $a$  since  $a$  by  $c$  is strictly greater than one

so this  $x$  equal to  $a^2$  by  $c$  lies on the right of this ellipse now if we take any point  $p(x, y)$  on the ellipse let us look at the distance  $p f_1$  and the distance to this line  $p l$

so distance of  $p(x, y)$  on the ellipse to  $f_1$  which is  $c$  zero is  $p f_1$  equal to square root of  $(x - c)^2 + y^2$

so  $p f_1$  square is  $(x - c)^2 + y^2$  which is equal to  $x^2 - 2cx + c^2 + y^2$  but what is  $y^2$   $y^2$  is equal to  $b^2$  times one minus  $x^2$  by  $a^2$

so let us put this

so this gives say  $x^2 - 2cx + c^2 + y^2$  is  $b^2$  times one minus  $x^2$  by  $a^2$

so this gives one minus  $b^2$  by  $a^2$   $x^2 - 2cx + c^2$  plus  $b^2$  but what is this this is equal to  $a^2 - b^2$  by  $a^2$   $x^2 - 2cx + c^2 + b^2$  is equal to  $a^2 - b^2$   $c^2$  is given by  $a^2 - b^2$  again let us put  $a^2 - b^2$  as  $c^2$  here

so this is  $c^2$  by  $a^2$   $x^2 - 2cx + a^2$

so this is same thing as  $c$  by  $a$   $(x - a)^2$  right because if you square this you get this term plus  $a^2$  minus two times this will give two  $c$   $x$  here lets ah put  $c^2$  by  $a^2$  outside then we get  $x^2 - a^2$  by  $c$  whole square but what is the distance of  $p$  to  $l$  but if you see this point is  $p(x, y)$  this is  $l$  the distance from  $p$  to  $l$  is  $x - a$  square by  $c$  mod of this distance is mod of  $x - a$  square by  $c$

so therefore this  $x^2 - a^2$  by  $c^2$  is  $p l$  square

so thus we get  $p f_1$  square is equal to  $c$  by  $a$  times  $p l$  one square that is  $p f_1$  is  $c$  by  $a$  times  $p l$  one recall that we defined the eccentricity of the ellipse to be  $c$  by  $a$

so this is  $e$  times  $p l$  one  $p f_1$

so thus  $p f_1$  by  $p l$  is equal to  $e$

so the ratio of any point on the ellipse to the distance to this line  $x^2$  this ellipse and the line  $x$  equal to  $a^2$  by  $c$   $p f_1$  by  $p l$  is equal to  $x$  equal to the constant  $e$  similarly if we take this line  $x$  equal to minus  $a^2$  by  $c$  similarly for the line  $x$  equal to minus  $a^2$  by  $c$  the ratio is equal to  $e$

so thus these lines  $x$  equal to plus minus  $a^2$  by  $c$  are called the directrices of the ellipse  $x^2$  by  $a^2$  plus  $y^2$  by  $b^2$  equal to one with  $a$  greater than  $b$  we can do conversely and now we can define ellipse or also hyperbola using directrix

so let's let us take a conic section we will see that we will get a parabola if hyperbola using this conic section having a vertex at the origin focus on the  $x$  axis and directrix a line parallel to the  $y$  axis

so we have  $x$  axis and  $y$  axis we take a vertex  $v(0, 0)$  let us take focus to be this point with coordinate  $f, 0$  and lets take a line  $l$  what is the equation of this line this line is  $x$  equal to some  $\alpha$  now let us look at let us find the locus of points  $p$  such that the ratio  $p f$  divided by  $p l$  where this is the perpendicular of any point

so if we take any  $p(x, y)$  we look at  $p f$  distance and this perpendicular distance  $p l$  this is equal to some constant  $e$  this is a constant then because we have  $v$  since  $v(0, 0)$  lies on the curve we have  $v f$  divided by  $v l$  this should be equal to  $e$  this implies  $v l$  distance is  $v f$  by  $e$   $v f$  is equal to  $f$  here

so this is equal to  $f$  by  $e$

so therefore the line  $l$  is  $x$  equal to minus  $f$  by  $e$  let us draw this again and we have this line  $x$  equal to minus  $f$  by  $e$  now take any general point  $p(x, y)$  on the curve then  $pf^2$  is equal to  $e^2 p l^2$  what is  $pf^2$   $pf^2$  square is  $(x - f)^2 + y^2$  this is equal to  $e^2$  times  $p l^2$  this length is  $x + f$  by  $e$

so  $(x + f)^2 + y^2 = e^2 (x + f)^2$

so if we simplify this this gives  $x^2 - 2fx + f^2 + y^2 = e^2(x^2 + 2fx + f^2)$  square equal to this is  $e^2 x^2 + 2e^2 fx + e^2 f^2$   $x^2 + y^2 - 2fx + f^2 = e^2 x^2 + 2e^2 fx + e^2 f^2$

so  $f^2$  cancels and this gives  $(1 - e^2)x^2 - 2fx + y^2 = 2e^2 fx + e^2 f^2$  plus  $e^2 x^2 + y^2 = 0$

so let us put  $p$  is equal to  $f$  times one plus  $e$  we get  $(1 - e^2)x^2 - 2fx + y^2 = 2e^2 fx + e^2 f^2$  square minus  $2px + y^2 = 0$  now if we look at this equation this is the equation such that the ratio of the distance of  $p$  to  $f$  and  $p$  to the line  $l$  is a constant  $e$  now if  $e$  is equal to one we get and the first term is not there we get  $y^2 = 2px$  which is a parabola

so this we already saw before that if we take the ratio to be equal to one then we get a parabola what happens if  $e$  is less than one if  $e$  is less than one then  $1 - e^2$  is positive this is the positive

so we get some  $a^2 x^2 - 2px + y^2 = 0$

so this we can write as  $x^2 - 2px/a^2 + y^2/a^2 = 0$  or  $(x - p/a)^2 + y^2/a^2 = p^2/a^4$  square equal to zero or  $(x - p/a)^2 + y^2/a^2 = (p/a^2)^2$  square equal to  $p^2/a^4$  which is equation of an ellipse though it is not in the standard form but here the ellipse you see if you put  $y$  equal to zero then you get  $(x - p/a)^2 = p^2/a^4$  square is  $p^2/a^4$  square

so  $x$  is equal to the center of the ellipse is at the point  $p/a^2$  and zero and we get an ellipse like this in the special case equal to zero if you see this was the general equation if I put  $e$  equal to zero gives  $x^2 - 2px + y^2 = 0$  which is equation of a circle

so in ellipse we see if we put  $e$  equal to zero we get equation of a circle and if  $e$  is greater than one then we can write the equation as this equation if  $e$  is greater than one this is negative

so we will write  $e^2 x^2 - 2fx + y^2 = 2e^2 fx + e^2 f^2$  square equal to zero and this here is positive

so this is an equation of a hyperbola

so one can define this ellipse and hyperbola also using focus and directrix right

so all these conic sections parabola ellipse and hyperbola can be defined as the locus of all points such that the ratio of for the distance of the point to a fixed point called focus and a fixed line is a constant  $e$  which is the eccentricity ok

so so next just like we did for ellipse we found the equation of the directrices similarly for hyperbola there will be two directrix and the equation of the directrix we can find

so consider the hyperbola  $x^2/a^2 - y^2/b^2 = 1$

so here the vertices are  $(a, 0)$  and  $(-a, 0)$  and focus  $f$  one is  $(c, 0)$   $(-c, 0)$  there is another focus here  $f^2 - c^2 = 0$  now again let us look at the line  $x = a^2/c$  in the case of hyperbola we know that  $c^2 = a^2 + b^2$  squared is equal to  $a^2 + b^2$

so therefore this is greater than  $a^2$

so  $a^2/c^2 < 1$

so therefore  $a^2 < c^2$  this is strictly less than  $a^2$

so this line here is  $x = \pm \frac{a^2}{c}$  now if i take any point  $P(x, y)$  let us calculate what is the distance  $PF_1$  and this distance  $PL$

so  $PF_1^2$  is equal to  $(x - c)^2 + y^2$  which is equal to  $x^2 - 2cx + c^2 + y^2$  but  $y^2 = \frac{b^2}{a^2}(x^2 - \frac{a^2}{c^2})$  so  $PF_1^2 = x^2 - 2cx + c^2 + \frac{b^2}{a^2}(x^2 - \frac{a^2}{c^2})$

so therefore  $PF_1^2 = x^2 - 2cx + c^2 + \frac{b^2}{a^2}x^2 - \frac{b^2}{c^2}$

so this is equal to  $(1 + \frac{b^2}{a^2})x^2 - 2cx + c^2 - \frac{b^2}{c^2}$  but  $a^2 + b^2 = c^2$

so this is  $\frac{c^2}{a^2}x^2 - 2cx + c^2 - \frac{b^2}{c^2}$  so  $PF_1^2 = \frac{c^2}{a^2}(x^2 - 2ax + a^2) - \frac{b^2}{c^2}$

so this can be written as  $\frac{c^2}{a^2}(x - a)^2 - \frac{b^2}{c^2}$  which again i can put  $c^2 = a^2 + b^2$

so  $\frac{c^2}{a^2}(x - a)^2 - \frac{b^2}{c^2} = \frac{a^2 + b^2}{a^2}(x - a)^2 - \frac{b^2}{c^2}$  since  $\frac{c}{a}$  is eccentricity  $e$  this is  $e^2 PL^2 - \frac{b^2}{c^2}$

so therefore  $PF_1 = e PL$  which is greater than one in the case of hyperbola

so the lines  $x = \pm \frac{a^2}{c}$  are the directrices of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  if we take the hyperbola with foci on the  $y$  axis if we take  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$  then the directrices will be  $y = \pm \frac{a^2}{c}$

so so again just like an ellipse we get equation of the directrices given by  $x = \pm \frac{a^2}{c}$  and  $x = \pm \frac{a^2}{c}$  in this standard form

so this shows that all the conic sections parabola ellipse and hyperbola can be described in terms of focus and directrix and the definition is that the ratio of the distance to a fixed point focus and the perpendicular distance to a line called directrix should be a constant and depending on whether the constant is equal to 1 less than 1 or greater than one we get parabola ellipse and hyperbola

so we will stop here today and in the next class we will learn about tangents and normals of parabola ellipse and hyperbola etc thank you