

hello everyone

so this is the fourth lecture on conic sections in the first three lectures we learned the standard equations of parabola ellipse and hyperbola and looked at some simple problems

so let us continue our discussion

so let us recall standard equation of a hyperbola is of the form $x^2/a^2 - y^2/b^2 = 1$ or $y^2/a^2 - x^2/b^2 = 1$

so this is first and this is second form the first form this the graph of this hyperbola looks like it has two branches and this center is at the origin the vertices of the hyperbola are at point $a, 0$ and $-a, 0$ this is obtained by putting y equal to zero in equation one we get $x^2/a^2 = 1$

so x is plus minus a and the focus the two focus

so let us call f_1 and f_2 these have coordinate $c, 0$ and $-c, 0$ where $c^2 = a^2 + b^2$ this is the first one and the second form here if you put x equal to zero then we get $y^2/a^2 = 1$

so vertices are at point $0, -a$ and $0, a$ and the hyperbola looks like this and here again the foci are at the point $0, c$ and $0, -c$ and again $c^2 = a^2 + b^2$

so we looked at two examples in the last class let's do some more problem

so find the equation of the hyperbola whose vertices are at plus minus two zero and foci at plus minus three zero

so since the vertices are on the x axis since the vertices are on the x axis the equation of the hyperbola is of the form $x^2/a^2 - y^2/b^2 = 1$ here it is given that vertices are at plus minus two zero vertices of this hyperbola are at plus minus $a, 0$ and foci are plus minus $c, 0$

so in the given problem a is equal to two and c is equal to three in this equation we need the value of a and b

so we have $a^2 + b^2 = c^2$ this implies $b^2 = c^2 - a^2$ which is equal to 5 and therefore the equation is $x^2/a^2 - y^2/b^2 = 1$ that is $x^2/4 - y^2/5 = 1$ now let's look at second problem here you are given that vertices are at zero plus minus five and foci at zero plus minus eight

so in this the equation will be of the form $y^2/a^2 - x^2/b^2 = 1$ because the vertices are on the y axis the equation is of the form $y^2/a^2 - x^2/b^2 = 1$ where a is given to be five and c is given to be eight

so from this we can again find b therefore $b^2 = c^2 - a^2$ which is $8^2 - 5^2$ that gives 39 and therefore the equation is $y^2/a^2 - x^2/b^2 = 1$ let us look at one problem where you are given foci in the length of lattice rectum find the equation of the hyperbola whose foci are at plus minus four zero and the length of the lattice rectum is twelve

so again since foci are on the x axis on the x axis the equation is $x^2/a^2 - y^2/b^2 = 1$ and foci at plus minus four zero this implies $c = 4$ that is $a^2 + b^2 = 16$ recall that length of the lattice rectum length of lattice rectum l is given by $2b^2/a$ this is given to be equal to twelve i have this one equation and then $2b^2/a = 12$ this implies $b^2 = 6a$ putting this in equation one we get $a^2 + 6a = 16$ this is $a^2 + b^2 = c^2 = 16$

so therefore this is equal to sixteen and this implies a square plus six a minus sixteen equal to zero you can easily factorize this as a minus two times a plus eight equal to zero since a is positive this implies a is equal to two

so a is two therefore you can calculate b therefore b square is equal to six a which is equal to twelve this implies we need b square and a square

so therefore the equation is x square by a square is four minus y square by twelve equal to one let me do one more simple problem you have to find the equation of the hyperbola whose foci are at 0 plus minus root 10 and which passes through the point two comma three

so foci is given to be zero plus minus root ten this implies c is equal to root ten also the equation is because the foci lies on the y axis equation is of the form y square by a square minus x square by b square equal to one and since this equation passes through since it passes through 2 comma 3 we get three square nine by a square minus two square by b square four by b square this is equal to one this is equation one and second equation we get in terms of a and b by using c is equal to root ten also a square plus b square equal to c square which is ten this is my equation two and using 1 and 2 we can solve for a n a square and b square

so from here we can write b square is equal to ten minus a square and putting this in equation one gives nine by a square minus four by ten minus a square equal to one which implies 9 times 10 minus a square minus 4 a square equal to a square times 10 minus a square which implies ninety minus nine a square minus four a square equal to ten a square minus a to the fourth this gives a quadratic in a square

so this gives a to the four minus we have ten here and nine plus four is thirteen twenty three a square plus ninety equal to zero and now you can find a square from here

so this implies we can write this as a square minus five times a square minus eighteen equal to zero this implies a square equal to five or a square is equal to eighteen but a square plus b square this is given to be ten this implies a square is less than equal to ten hence a square must be equal to five and then using a square plus b square equal to ten this gives b square is also five therefore the equation is y square by b y square by a square is five minus x square by b square five equal to one

so in this case a and b are equal that is same thing as y square minus x square equal to five ok

so so far we have looked at the standard equation of hyperbola and then seen some problems on finding the equation of hyperbola given vertices foci or some points on the hyperbola etcetera now let let me talk about what is called asymptotes of a hyperbola

so consider the hyperbola x square by a square minus y square by b square equal to one

so if we draw this hyperbola this has vertices at a zero and minus a zero foci are at the point c comma zero minus c zero and the graph it looks like this passes through these vertices now let us look at

so we have a and c are the coordinates of these this vertex and this focus now where is b what we have is a square plus b square equal to c square

so if i this is the origin if i draw this circle centered at the origin and radius equal to c i get this circle now what we have a square plus b square equal to c square

so if i draw a right angle triangle where this is length is a and then the perpendicular is here this length is the radius of the circle is c

so you can see here that this height of this right angle triangle is given by b so b is this length now let us look at this straight line

so this is the straight line whose slope is b by a and we can look at another straight line whose slope is $-b$ by a

so if i do the same thing here i have again this length is c and this is a therefore this will be b

so this straight line has slope $-b$ by a

so let me mention this this is the line y equal to b by a x and this line is y equal to $-b$ by a x now let's first see whether these straight lines whether they intersect with the hyperbola or not

so if we see the straight lines y equal to $\pm b$ by a x do not intersect the hyperbola x^2 by a^2 minus y^2 by b^2 equal to one because if y is $\pm b$ by a x that is if there is a point lying on this one of these straight line this will imply that y^2 by b^2 is same as x^2 by a^2 which will imply that x^2 by a^2 minus y^2 by b^2 is equal to one hence x^2 by a^2 minus y^2 by b^2 is equal to zero but the equation of the hyperbola is x^2 by a^2 minus y^2 by b^2 equal to one

so any point on these straight lines x comma y does not lie on the hyperbola

so these lines they do not intersect the hyperbola however if you look at the line y equal to b x and this hyperbola as x tends to infinity we can show that the line and this hyperbola they both tend to the same

so what i want to say is however if you look at the any x comma y on the hyperbola if x comma y lies on the hyperbola and x comma y one lies on the line say y equal to b by a x then what is y minus y one

so y lies on the hyperbola

so y can be written as b times square root of x^2 by a^2 minus 1 and y one is on the a line

so y one is b by a x actually if you see if i take any e x here and this is my x comma y and this is my x comma y one

so this difference in the y coordinate is actually y one minus y

so let me take mod of this which is equal to b by a and then we have x minus square root of x^2 minus a^2 now what happens if you have done calculus you can try to find what happens as x tends to infinity

so as x tends to positive infinity what is limit of x going to infinity of x minus square root of x^2 minus a^2 to evaluate this limit we can write this as limit x tends to positive infinity you multiply and divide by the conjugate

so x plus square root of x^2 minus a^2 times x minus square root of x^2 minus a^2 divided by x plus square root of x^2 minus a^2 and then the numerator is x^2 minus x^2 minus a^2 divided by x plus square root of x^2 minus a^2 which is equal to limit x tends to infinity of a^2 divided by x plus square root x^2 minus a^2

so the numerator is finite here denominator is x plus square root x^2 minus a^2 this goes to infinity

so therefore this goes to zero therefore the line y equal to b by a x and hyperbola x^2 by a^2 minus y^2 by b^2 equal to one approaches each other as x tends to positive infinity similarly you can do for the other line and taking x tends to negative infinity

so these lines are therefore the asymptotes of the hyperbola

so the line y equal to b by a x and y equal to $-b$ by a x these two are asymptotes now one definition if a is equal to b then the hyperbola is called a rectangular hyperbola or sometime it is called equilateral also

so rectangular or equilateral hyperbola this is because in this case the asymptotes are y equal to $\pm b$ by a x and these are perpendicular to each other which are perpendicular to each other

so in this case we have $x^2/a^2 - y^2/b^2 = 1$ with $b = a$

so we have $x^2/a^2 - y^2/a^2 = 1$ which implies $x^2 - y^2 = a^2$

so if you draw this hyperbola then here these lines $y = x$ and $y = -x$ these are the asymptotes and we have this vertex $(0, a)$ and the hyperbola will have these lines as the asymptotes and in this case say the vertices are at the point $(0, a)$ and $(0, -a)$ because $b = a$ the focus will be at the point $(0, \sqrt{2}a)$ and $(0, -\sqrt{2}a)$ right because $a^2 + b^2 = c^2$

so c is this is f_1 and f_2 the foci ok

so we have this rectangular hyperbola $x^2 - y^2 = a^2$ this can be written as $(x - y)(x + y) = a^2$ and in this case what we saw is that we have $y = x$ and $y = -x$ these are the asymptotes which are perpendicular to each other now if we change make a change of variable and put

so putting say $x' = x - y$ and $y' = x + y$ we get $x' y' = a^2$

so that means that what we are doing instead of taking this axis as x axis and y axis suppose i take this as x' and this as y' then what we get is this or let us call this axis as x' and this is y' then we have $x' y' = a^2$

so this gives another form of rectangular hyperbola rectangular hyperbola that is $x' y' = a^2$ this is more standard form of a rectangular hyperbola if you draw this hyperbola now the asymptotes will be x' and y' axis and the hyperbola is like this right

so this this part is $y' = a^2/x'$ for x' positive and this is for x' negative

so here the graph is in the first and third quadrant and the asymptotes are x' axis and y' axis asymptotes are x' axis and y' axis ah lets say as a special case if a is equal to one we get $x' y' = 1$ or $y' = 1/x'$

so you might be familiar with the graph of this function $y = 1/x$ this looks like this the graph is in the first quadrant and the third quadrant it is defined for all x non zero and as x goes to 0 infinity this goes to zero and x goes to negative infinity again it goes to zero as x goes to zero from the right hand side then y approaches plus infinity as x goes to zero from left hand side y approaches minus infinity

so now let us try to find let us find the vertices and foci of the rectangular hyperbola $x' y' = a^2$

so we have this rectangular hyperbola clearly here the center is at the origin

so this is the center O and the vertices of this hyperbola will lie on this line $y' = x'$

so here we have this is the transverse axis and the line $y' = -x'$ is the conjugate axis the vertices suppose the coordinate of this vertex is (a, a) then this vertex will be $(-a, -a)$ this is because this vertex x and y coordinates are same

so $x' y' = a^2$

so we get a and a

so vertices are (a, a) and $(-a, -a)$ what about the foci

so we know that foci will lie somewhere here let us say (c, c) and $(-c, -c)$

so let foci be (c, c) and $(-c, -c)$ then what we know is that for this rectangular hyperbola a is equal to b

so since $a = b$ for rectangular hyperbola we have $c^2 = 2a^2$

two a^2 which means c is equal to $\sqrt{2}a$

so the foci are at the point $(\sqrt{2}a, \sqrt{2}a)$ and $(\sqrt{2}a, -\sqrt{2}a)$ right

so we have found the

so foci $(\sqrt{2}a, \sqrt{2}a)$ and $(\sqrt{2}a, -\sqrt{2}a)$ we can also obtain this by using the basic definition of hyperbola

so let us find the foci of $xy = a^2$ using the definition of hyperbola

so we have this hyperbola $xy = a^2$ center is at the point $(0, 0)$ and because the hyperbola is symmetric about the transverse and conjugate axis we have this line $y = x$ as the transverse axis and $y = -x$ is the conjugate axis

so the vertices are at the point (a, a) and $(a, -a)$ suppose the foci are at the point (c, c) and $(c, -c)$ this foci also lies on the transverse axis

so we get this now if i take any point p

so by the definition of hyperbola if p is any point on the hyperbola say p lying in the first quadrant then $|p f_2| - |p f_1|$

so the difference of the distance of any point from the foci is constant two way right

so $|p f_2| - |p f_1|$ is equal to $2a$ now we can take the point

so let p be the point whose x coordinate is c

so if we take say this point p is x coordinate is c then y what is y coordinate because $xy = a^2$ this will be a^2/c

so p is $(c, a^2/c)$ now if i calculate the distance of this point to f_2

so $|p f_2|$ will be $\sqrt{(c-c)^2 + (a^2/c - c)^2}$ and $|p f_1|$ is equal to $\sqrt{(c-c)^2 + (a^2/c + c)^2}$ because this same x coordinate

so $|p f_1|$ is this

so i think we made a mistake here

so the difference of any point $|p f_2| - |p f_1|$ this must be constant but the constant is equal to the distance between the two vertices if i call this a and b this is not $2a$ this should be equal to $2b$ which is equal to $2\sqrt{c^2 - a^2}$ so $\sqrt{(c-c)^2 + (a^2/c - c)^2} - \sqrt{(c-c)^2 + (a^2/c + c)^2} = 2\sqrt{c^2 - a^2}$ this implies $\sqrt{c^2 + (a^2/c - c)^2} - \sqrt{c^2 + (a^2/c + c)^2} = 2\sqrt{c^2 - a^2}$ square plus four a^2/c times c minus a^2/c and this gives $4c^2 + 4a^2 - 4ac$ square plus four a^2/c times c minus a^2/c and this gives $4c^2 + 4a^2 - 4ac$ square plus four a^2/c times c minus a^2/c and then solving this will get c is equal to $\sqrt{2}a$ from here

so therefore the foci are at the point $(\sqrt{2}a, \sqrt{2}a)$ and $(\sqrt{2}a, -\sqrt{2}a)$

so this finishes today's lecture in the next lecture we will learn something more about parabola ellipse and hyperbola thank you you