hello everyone

so this is the third lecture on conic sections in lecture one we studied about parabola and then derived the standard equations of parabola and then discussed some properties in the second lecture we discussed about ellipse and learnt the standard equations of ellipse now in this lecture we will study about ah third kind of conic sections which is called hyperbola

so let us start with what is in a hyperbola

so the definition is a hyperbola is the set of all points in a plane the difference of whose distances from two fixed points in the plane is a constant and will take the constant to be less than the distance between the two fixed points

so recall that we define ellipse to be the set of all points in a plane the sum of whose distances from two fixed point is a constant here the difference is that we instead of sum of the distances from two fixed point we take the difference of the distances from two fixed point

so just like for ellipse these two the two fixed points are called the foci of the hyperbola also the midpoint the midpoint of the line segment joining the foci is called the center center of the hyperbola and the line the line through the two foci is called the transverse axis of the hyperbola and the line which is perpendicular to transverse axis

so the line perpendicular to the transverse axis and passing through the center is called the conjugate axis of the hyperbola

so let let me draw this

so we have two fixed points in the plane let us call them f1 and f2

so these are both foci

so this is a focus this is another focus of the hyperbola the midpoint of the line segment joining these two foci is called the center

so this is the center let us say o this is the center of the hyperbola and the line which passes through these two foci this line will be called the transverse axis and the line perpendicular to the transverse axis and passing through the center this is the conjugate axis

so now let us assume that the distance from the center to each of the foci is c

so this distance is c this is c

so let the distance between the two foci that is f one f two is equal to two c this is a positive number

so and let the center be at the origin and the foci lie on the x axis

so we have this is the x axis and we are taking the center to be the origin and foci f 1 and f 2 lie on the x axis

so x axis is the transverse axis and the perpendicular will be y axis which is the conjugate axis and what we have is that the distance between f one f two is two c

so these distances are c and c

so therefore the coordinates of f one f two is this is minus c comma zero this is c comma zero

so then f one is minus c zero f two is c comma zero now we want to trace this hyperbola

so hyperbola is the set of all points such that the difference of the distance from f one and f two is zero

so let us say there is one point p here then we have p f one is the larger distance here p f 2 is the smaller distance from if ok

so p f 1 minus p f 2

so if p is any point on the hyperbola then p f one minus p f two this is in absolute value equal to constant let us say this constant is equal to 2 a

so then we have in the definition that this constant is less than the distance

between the two foci
 so we have 2a is less than 2c
 so a is less than c now let us try to find some points on this hyperbola
 so suppose there is a point a on this hyperbola
 so let a whose coordinate is x comma zero b a point on the hyperbola
 so let us draw this again f1 f2 this is minus c comma zero this is c comma zero now if we have a point x comma zero
 so if x is between 0 and c then we see that this distance lets say this is the point a a f 1 minus a f 2 this is the difference between the distance of the point a to the foci f one and f two this we want to be equal to two a now what is a f one a f one is this distance is c this is x
 so a f 1 is equal to c plus x and a f 2 is equal to c minus x because this is again c
 so a f 1 minus a f two is c plus x minus c minus x which is equal to two x but what we want is that a f one minus a f two is equal to two a
 so two x is equal to two a which implies x is equal to a similarly
 so what we get is a comma zero lies on the hyperbola similarly we can see that minus a comma zero also lies on the hyperbola also if mod x is greater than equal to c then x comma zero does not lie on the hyperbola this is because if you see if my x comma 0 is here then the distance of this to f 1 minus the distance of this 2 f 2 is nothing but the distance between f 1 and f 2.

 so this is because if x is greater than c or x is less than greater than equal to c or x is less than equal to minus c then the difference of distances from f one and f two is f one f two which is equal to two c this is not equal to two a right
 so if we have f one f two like this and if we have point a here x comma zero then a f one minus a f two is equal to f one f two similarly if i have a point b here then b f two minus b f one is again equal to f one f two
 so we have exactly two points on the x axis which lie on the hyperbola
 so there are exactly two points on the transverse axis which lie on the hyperbola the coordinates of these two points are minus a comma zero and a comma zero
 so these two points be called the vertices of the hyperbola
 so in the picture we have focus f one f two lying on the x axis minus c zero and c zero and we have these two points let us call them a and b whose coordinates are minus a zero and a zero these are the vertices these are called the vertices of the hyperbola now lets find the standard forms of hyperbola
 so there are two types one is foci on the x axis and second type is foci on the y axis
 so right now we are not discussing the general form we are taking these two forms
 so the first form that we have been discussing now we have the transverse axis is x axis and conjugate axis is y axis foci f one f two this is the center this is minus c comma zero f two has the coordinate c comma zero let us take p to be any point
 so let p x comma y be any point on the hyperbola with x greater than zero
 so we are taking the point to lie in the first quadrant or fourth quadrant
 so if i have any point p here whose coordinate is x comma y then we have p f one is the bigger distance p f two is the smaller distance from these two foci
 so then we must have p f one minus p f two is equal to constant which is two a
 so what is p f one p f one is the distance of the point x comma y to minus c comma zero
 so this is x plus c square plus y square under square root p f two is x minus c

square plus y square square root

so we have square root of x plus c square plus y square minus square root of x minus c square plus y square this is equal to constant which is two a now we have to find a equation between x and y

so we write x plus c square plus y square this will be two a plus square root of x minus c square plus y square whole square which is equal to four a square plus x minus c square plus y square plus four a times square root of x minus c square plus y square this is x plus c square plus y square here you can cancel y square this implies four a times square root of x minus c square plus y square equal to x plus c whole square minus x minus c squared minus four a square but x plus c squared minus x minus c squared this is equal to four x c four c x minus four a square canceling four from both sides and then squaring we get a square times x minus c square plus y square this is equal to c x minus a square whole square which implies a square x square plus a square c square minus 2 a square c x plus a square y square this is equal to c square x square plus a to the four minus two a square c x

so we have this term cancels and now we can write this because a is smaller than c

so we write this as c square minus a square x square minus a square y square this is equal to a square c square minus a to the four which is equal to a square times c square minus a square

so lets put c square minus a square equal to b square then we get b square x square minus a square y square is equal to a square b square dividing by a square b square we get x square by a square minus y square by b square is equal to one

so this is the equation which is satisfied for any point x with x positive

so we see from here that this will imply x square by a square equal to 1 plus y square by b square

so if we take any y this is always greater than or equal to 1 which implies x square is greater than equal to a square

so this implies x is greater than equal to a for x lying in the positive half plane

so any point on this hyperbola we have these are for chi f1 f2 and we have two vertices minus a comma zero a comma zero also from this equation you can see that if y is zero then x square equal to a square

so a comma zero is a point here and this says that x is always greater than equal to a

so any point will lie on the right of the line x equal to a this is line x equal to a

so if p is a point here this is x comma y

so if you trace this you will get something like this also from here you can see that this equation is symmetric about x axis and y axis if i put if x comma y lies on this then x comma minus y this will also lie

so the graph will look like this similarly you can show that for x negative again we get the same equation like this

so similarly if p is x comma y with x less than zero on the hyperbola then

so now if p is any point here we have p f two is bigger than p f one

so in this case we will solve for p f two minus p f one is equal to two a and proceeding like in the previous case we get the same equation x square by a square minus y square by b square equal to one where b square is c square minus a square

so in this case again we have a point here minus a comma 0 which lies on the hyperbola and the hyperbola will be like this

so now we get the graph

so the hyperbola x square by a square minus y square by b square equal to one looks like we have two vertices minus a comma zero a comma zero the hyperbola passes through these two vertices it is always for x greater than a or x less than minus a and it will be symmetric about the x axis also symmetric about the y axis

so this hyperbola has two branches one is this one for x positive and this is for x negative this is symmetric the hyperbola is symmetric about the transverse axis as well as they conjugate axis now second form of hyperbola is when the foci on the y axis

so this is the second form foci on the y axis

so in this case your x axis y axis foci lies on the y axis

so the coordinates will be zero minus and 0 c and now we have this y axis in this case is the transverse axis and this is the conjugate axis and we can see that the graph that we will get is by simply interchanging this x and y axis

so the hyperbola will look like this right and this is the point zero comma a this is zero minus a the equation of the hyperbola with foci on the y axis is given by we replace x and y

so we have y square by a square minus x square by b square equal to one

so this is the hyperbola and you can see that this intersects the y axis at plus minus zero sorry this intersects the y axis at zero plus minus a and this will not intersect the x axis because if you put y equal to zero we get minus x square by b square equal to one which does not have any real root also mod y is always greater than equal to a that is y is either greater than equal to a or y is less than equal to minus a

so now just like for ah ellipse we will define lattice rectum of hyperbola

so let us look at the hyperbola with transverse axis as x axis these are vertices minus a zero and a zero and the focus here is c comma zero and another focus is at minus c comma zero

so what is lattice rectum lattice rectum is the line segment joining two points on this hyperbola such that this passes through one of the foci and is perpendicular to this transverse axis

so so this is like one lattice rectum another will be on the negative x axis like this

so this is line segment passing through through a focus and perpendicular to the transverse axis and having end points on the hyperbola

so we would like to find the length of this lattice rectum

so if i call this point a this point as b then we can see by symmetry that if both has x coordinate as c and if y coordinate is beta here then this is minus beta

so let a equal to c minus beta and b equal to c beta then we want to find the length

so length of the lattice rectum then length of the lattice rectum l is equal to 2 beta and this is same thing as length of this other lattice rectum by symmetry

so we want to find what is beta

so since c comma beta lies on the hyperbola whose equation is x square by a square minus y square by b square equal to one we get you put x equal to c c square by a square minus beta square by b square this is equal to one this implies beta square by b square is c square by a square minus one which is c square minus a square by a square but c square minus a square we have called it b square

so this is b square by a square this implies beta square is b to the four by a square which implies beta is b square by a

so length of the lattice rectum l is two times b square by a this is the same formula as we got for ellipse length of lattice rectum is two b square by a now

just like for ellipse we define the eccentricity of the hyperbola as e is equal to c by a

so for hyperbola what we see is that a is strictly less than c

so this is greater than one now we will look at a few problems to find the foci vertices eccentricity and length of the lattice rectum for let us look at first hyperbola as x square by 16 minus y square by nine equal to one and second one is nine y square minus four x square equal to thirty six

so the first problem if we see we have x square by 16 minus y square by nine equal to one this is of the form x square by a square minus y square by b square equal to one

so this says a is equal to four b is three and foci lie on the x axis

so foci the coordinate is plus minus c comma zero what is c in this case c square minus a square equal to b square this implies c square is a square plus b square which is four square plus three square this is twenty five this implies c is equal to five

so foci are at plus minus five comma zero vertices at plus minus a zero

so this is plus minus four zero eccentricity e is equal to c by a c is five and a is four

so this is five by four and the length of lattice rectum l is equal to two b square by a which is equal to two times b is three here

so nine by a is four this is nine by two similarly for the second problem we have nine y square minus four x square equal to 36

so we write first in the standard form this means y square by 4 minus x square by 9 is equal to 1 this is of the form y square by a square minus x square by b square equal to one with a equal to two and b equal to three

so this form is when the foci lies on the y axis

so therefore foci will lie on the y axis and and will have coordinate zero plus minus c what is c c square is again a square plus b square which is two square plus three square four plus nine is thirteen

so c is square root of thirteen

so therefore foci have coordinates zero plus minus square root thirteen vertices are now on the y axis which coordinate zero plus minus a a is equal to two

so this is zero plus minus two eccentricity e is c by a c is root thirteen divided by a is two and length of lattice rectum l is two b square by a that is two times b is three here three square by a is two

so this is equal to nine ok

so we will end the this lecture here in the next lecture we will discuss some more problems on hyperbola and then we will discuss some more advanced topics on parabola ellipse and hyperbola thank you you