

welcome to the second lecture on conic sections

so in the first lecture we discussed about parabolas now in this lecture we will talk about ellipse which is generalization of circle

so let us first define what do we mean by ellipse

so the definition is an ellipse is the set of all points in a plane that are an ellipse is the set of all points in a plane such that the sum of the distances from two fixed points points in the plane is a constant

so what we have is we have two fixed points let's call them  $f_1$  and  $f_2$  and then we are looking for all the points in this plane such that if we look at the distance of the point let us say  $p$  is a point then we have the sum of the distance of this point from these two fixed points  $f_1$  and  $f_2$

so we have  $pf_1 + pf_2$  this is a constant ok

so a special case as if these two points are same

so if  $f_1$  is equal to  $f_2$  then what do we get

so we have only one point and then we are looking for this sum of the distance of points from these two points which are identical

so we have  $pf_1$  is equal to  $pf_2$

so we have only one point ah  $f$  this is  $f_1$  as well as  $f_2$  and then if i take any point  $p$  the sum of the distance of this point from  $f_1$  and  $f_2$  is twice the distance of this point  $p$  to  $f$

so if this distance we call  $r$  then  $pf_1 + pf_2$  is equal to  $2r$  which is a constant

so what we get is we get a circle if these two points coincide then we get a circle what do we get if these two points are not the same

so in

so ellipse is a special case of circle if i have now this point  $f_1$  and  $f_2$  and then we look at all the points such that the sum of the distance from  $f_1$  and  $f_2$  is constant we get a curve like this

so if i take any point  $p$  this plus this is same as this plus this same as this plus this for any point  $p$  if you take  $p_1, p_2, p_3, p_4, p_i$   $pf_1 + pf_2$  is a constant

so just like to draw a circle you can take a fixed point center of the circle and then now you take a fixed radius  $r$  and you can take a thread join from here and then if you just rotate it about this you get a circle in this case what you can do is if you take two different fixed point take a thread and then take this at some point and if you just keep the length of the thread same

so or you keep rotating you get this ellipse

so some terms that we will use these two fixed point the two fixed points are called the foci of the ellipse foci is plural of focus

so there are two foci of the ellipse the midpoint between the two foci this is called the center

so center this is the midpoint on the line segment joining the foci also if i look at the line passing through these two foci this is called let me call this  $a$  and  $b$  this is called the major axis and the line segment which is perpendicular to the major axis and passing through the center this is called the minor axis major axis this is the line segment joining two points on the ellipse passing through the foci and minor axis is the line segment passing through the center and perpendicular to the major axis also we define vertices of an ellipse these are the end points of the major axis

so let us draw an ellipse we have this is the center these are the foci  $f_1$  and  $f_2$  let's write this  $a, b$  and  $c, d$

so  $a$  and  $b$  are vertices this is  $a, b$  is the major axis  $c, d$  is the minor axis  $f_1$  and  $f_2$  are foci and let us call this point  $o$  as the center

so  $o$  is the center ok

so now let us consider an ellipse whose foci lie on the x axis and let us say the center is at the origin

so we have origin is here and x axis y axis because the center is the midpoint of the foci and the foci lie on the x axis if we write this  $f_1$  and  $f_2$  the distance of this to the center  $o$  is same

so if the coordinate of  $f_2$  is  $c, 0$  then  $f_1$  will be  $-c, 0$  and let us say the vertices are vertices will also be on the x axis

so say the vertices  $a, b$  these are coordinates  $-a, 0$  and  $a, 0$  and let us say the minor axis we have this is  $0, b$  and  $0, -b$

so what is the relationship between  $a, b$  and  $c$   $a$  is the length of the half of the length of the minor axis  $b$  is half of the length of the major axis this is  $a$  this is our  $b$  and  $c$  is this distance

so now by the definition of ellipse we know that any point on the ellipse the sum of the distance from the foci  $f_1$  and  $f_2$  must be constant

so if we look at this point  $b$  what is the distance  $b, f_1$  plus  $b, f_2$  what is  $b, f_1$   $b, f_1$  is equal to  $b, 0$  plus  $0, f_1$  plus  $b, f_2$  is  $b, 0$  minus  $0, f_2$  now what is the distance of  $b, 2, 0$  this is  $a, 0$   $f_1$  is  $c$

so  $a + c$  plus  $b, 0$  is again  $a, 0$   $f_2$  is  $c$

so  $a - c$

so  $c$  cancels and this is equal to  $2a$

so  $b, f_1$  plus  $b, f_2$  is equal to  $2a$

so similarly lets calculate what is the distance

so lets call this point  $c$  and  $d$  what is  $c, f_1$  plus  $c, f_2$   $c, f_1$  is equal to if i draw this  $c, f_1$  is equal to square root of this  $c^2$  plus  $b^2$

so  $c, f_1$  is square root of  $b^2$  plus  $c^2$  and  $c, f_2$  is again same this is a square root of  $b^2$  plus  $c^2$  this is also square root of  $b^2$  plus  $c^2$

so therefore  $c, f_1$  plus  $c, f_2$  this is equal to twice of square root of  $b^2$  plus  $c^2$  now the definition of ellipse is that the sum of the distance of any points point from the two foci is a constant

so since  $b, f_1$  plus  $b, f_2$  should be equal to  $c, f_1$  plus  $c, f_2$  we get  $2a$  is this distance equal to  $2$  times square root of  $b^2$  plus  $c^2$  which implies  $b^2$  plus  $c^2$  equal to  $a^2$  or  $c^2$  is  $a^2$  minus  $b^2$

so  $c$  is square root of  $a^2$  minus  $b^2$

so this is the relation between  $a, b$  and  $c$  that if you have this semi major axis this length of the semi major axis and semi minor axis are  $a$  and  $b$  then the distance of any of the foci to the center  $c$  is given by  $c$  equal to square root of  $a^2$  minus  $b^2$  definition we define eccentricity of an ellipse this is  $e$  is equal to  $c$  by  $a$

so this is the ratio the ratio of the distance between the foci and the distance between the vertices right because say if you look at an ellipse in the foci the distance between the foci  $f_1, f_2$  is equal to  $2c$  and the distance between the vertices  $a, b$  is equal to  $2a$

so  $f_1, f_2$  divided by  $a, b$  this is  $2c$  over  $2a$  or equal to  $c$  over  $a$

so we can write  $c$  in terms of the eccentricity

so  $c$  is equal to  $a$  times  $e$  note that since  $c$  is smaller than  $a$  eccentricity will be less than one next we will try to derive a formula for the standard ellipse

so so let us take an ellipse with center at the origin and foci on the x axis

so we have we have the origin and then we have this is  $a, 0$  minus  $a, 0$  then the foci  $f_1, f_2$  is  $-c, 0$  and  $c, 0$  suppose  $p, x, y$  is any point on the ellipse ok

so we know that  $p, f_1$  plus  $p, f_2$  is a constant and we have already

calculated in previous slide if we take this point b if this is a and b then b f one plus b f two this we calculated is equal to two a

so p f one plus p f two must be equal to two a for every point p on the ellipse now what is p f one is  $x + c$  square plus y square square root plus p f two is  $x - c$  square plus y square square root equal to two a now from here we will derive an equation

so this is same thing as  $x + c$  square plus y square equal to two a minus square root of  $x - c$  square plus y square square which implies we get  $x + c$  square plus y square equal to four a square plus  $x - c$  square plus y square minus four a times square root of  $x - c$  square plus y square

so let us take the square root one side this means  $4 a$  times square root of  $x - c$  square plus y square this is equal to see y square cancels

so  $4 a$  square plus  $x - c$  square minus  $x + c$  square this is equal to minus  $4 x c$

so we get  $4 a$  times square root of  $x - c$  square plus y square equal to  $4$  times a square minus  $c x$  we can cancel  $4$  and then let us square both sides we get a square times  $x - c$  square plus y square is equal to a square minus  $c x$  whole square which is a to the four minus  $2 c a$  square  $x + c$  square  $x$  square

so we get a square  $x$  square plus a square  $c$  square minus  $2 a$  square  $c x$  plus a square y square equal to a to the four minus  $2 a$  square  $c x$  plus  $c$  square  $x$  square we can cancel this term  $2 a$  square  $c x$  and then we get a square minus  $c$  square times  $x$  square plus a square times y square equal to a to the four minus a square  $c$  square which is a square times a square minus  $c$  square but we have seen the relation between a b and c which was  $c$  square is a square minus b square or a square minus  $c$  square will be b square

so we will write a square minus  $c$  square this is b square this is b square

so we get b square  $x$  square plus a square y square equal to a square b square now dividing by a square b square this gives  $x$  square by a square plus y square by b square is equal to one

so this we have obtained as the equation this is the equation of an ellipse whose vertices are at minus a zero and a zero and center of course in this case center at the origin zero zero and in this case the length of major axis is two a length of minor axis is two b

so we will get ellipse like this this is the case where a is bigger than b or we can have the major axis is smaller than the minor axis

so this is the equation for a less than b both are this equation  $x$  square by a square plus y square by b square equal to one now from this equation we can see that this is symmetric about both x and y axis also if a is equal to b then we get a circle

so circle is a special case of ellipse in which the major axis and minor axis are of the same length

so we will get  $x$  square by a square plus y square by a square equal to one that is  $x$  square plus y square equal to a square

so this is the circle centered at origin centered at origin and radius a now just like we define lattice rectum for parabola we will define that lattice rectum for ellipse

so this is line segment with end points on the ellipse passing through a focus and perpendicular to the major axis

so if we have an ellipse like this this is the center we have focus f one f two then let me draw the lattice rectum is drawn in the red here what is the length of this lattice rectum

so if we look at

so we have this point as a comma zero this is the origin this is one of the

focus  $c$ , zero now let us take this point  $p$  here then  $p$  will have coordinate  $(c, 1)$  and then this will be  $(c, -1)$  this is let us say  $f$  is this point  $q$

so let  $PF = QF = 1$  then we need to find length of lattice rectum is  $2l$  now the coordinates of  $p$  is  $(c, 1)$  what is  $c$  we have seen that  $c$  can be written in terms of eccentricity as  $ae$  since  $p$  lies on ellipse  $x^2/a^2 + y^2/b^2 = 1$  we get  $a^2e^2/a^2 + 1/b^2 = 1$

so this gives  $1/b^2 = 1 - e^2$  which is  $1 - e^2 = c^2/a^2$

so  $1/b^2 = a^2 - c^2/a^2$   $1/b^2 = b^2/a^2$

so this will imply  $1/b^2 = b^2/a^2$  or  $1 = b^2/a^2$  so  $2l$  is the length of the lattice rectum this is equal to  $2b^2/a^2$  this is length of lattice rectum

so length of lattice rectum for an ellipse  $x^2/a^2 + y^2/b^2 = 1$  is  $2b^2/a^2$  now let us discuss some problems find the foci vertices eccentricity and length of lattice rectum for the ellipse  $16x^2 + y^2 = 16$

so first we will write the equation in the standard form

so dividing by sixteen we get  $x^2 + y^2/16 = 1$

so I will write  $x^2/1 + y^2/16 = 1$

so this implies  $a = 1$  and  $b = 4$

so in this case  $a < b$

so the foci will lie on the  $y$  axis

so here the ellipse will look like this

so we have this is  $(0, -1)$  and  $(0, 4)$  and  $(0, -4)$

so the foci in this case will be on the  $y$  axis

so this will be  $f_1$  and  $f_2$   $f_1$  is  $(0, -c)$   $f_2$  is  $(0, c)$  and eccentricity in this case  $e$  will be ratio of  $c$  over here the major axis is say the  $y$  axis

so  $c/b$  and we have this  $c^2$  in this case will be  $b^2 - a^2$

so this is  $16 - 1$  this is fifteen

so  $c$  is square root of 15 and eccentricity  $e$  is  $c/b$  which is square root 15 over 4 foci

so therefore the foci are at  $(0, \pm c)$  and length of lattice rectum

so in this case the lattice rectum will be this

so we derived this formula where the foci was on the  $x$  axis and length of lattice rectum was  $2b^2/a^2$

so if you just change the  $x$  and  $y$  axis you will see that if the foci lie on the  $y$  axis length of lattice rectum will be  $2a^2/b^2$  which is  $2a^2/b^2$  here  $2a^2/b^2$  sorry  $b$  is 4

so  $1/2$ .

this you can directly also calculate here we have this ellipse foci is this point is  $(0, \sqrt{15})$  then if you want this point here this is my  $f$  this point  $p$  will be some  $(x, \sqrt{15})$  and we have  $x^2/1 + y^2/16 = 1$  which means  $16x^2 + y^2 = 16$  which means  $16x^2 + 16 = 16$  means  $x = 1/4$ .

so this  $x$  is  $1/4$  length of the lattice rectum will be this  $p, q, p, q$  is

two x which is one by two which we can get directly from using this formula also let us look at one more problem find the equation of the ellipse whose center is at zero zero major axis is on the y axis and passes through these two point three two and one six

so we have this figure the major axis is on the y axis

so i will write  $(0, a)$  and  $(0, -a)$  as the vertices this is the origin and this is  $(b, 0)$  and  $(-b, 0)$

so the equation is  $x^2/b^2 + y^2/a^2 = 1$  one note that here i have taken this as  $(b, 0)$  and  $(0, a)$

so now this is the equation of an ellipse whose center is at the origin and major axis on the y axis ellipse centered at  $(0, 0)$  and major axis on the y axis now we are given that this ellipse passes through two given points

so we can use that to find the values of a and b

so since it passes through points three two and one six we get using  $(3, 2)$  we get  $9/b^2 + 4/a^2 = 1$  this is one equation and it passes through the point one comma six

so  $1/b^2 + 36/a^2 = 1$  this is equation two

so from equation one and two we can find the values of a and b

so if you see if i do 9 times equation 1 minus equation 2 this will imply  $81/b^2 + 36/a^2 - 9/b^2 - 36/a^2 = 9 - 9$

so  $80/b^2 = 0$  this implies  $b^2 = 10$  and therefore if i put  $b^2 = 10$  one implies  $9/10 + 4/a^2 = 1$  which means  $4/a^2 = 1/10$

so  $a^2 = 40$

so  $b^2 = 10$  and  $a^2 = 40$  therefore the equation is  $x^2/10 + y^2/40 = 1$  ok

so we will stop here for this lecture in the next class we will learn about hyperbola and also some more problems on parabolas and ellipses thank you