

welcome to the thirteenth lecture on circles in the last lecture we had discussed the method of deriving the equation of family of circles

so one special class of family of circles is those family of circles which touch a fixed point

so let us say that we have a fixed point whose coordinates are  $x_1$   $y_1$  and let us say that there is this line straight line whose equation is  $y - y_1 = m(x - x_1)$

so this point  $x_1$   $y_1$  lies on this straight line

so we are going to find the equation of the family of circles which touch which touch this straight line at exactly this point only

so obviously there are infinitely many circles for example this could be one circle this could be another circle or this could be another one

so what is the general equation of all these type of circles

so that is the ah question that we are trying to address here

so for any such circle which touches let us say this particular circle which touches the straight line at  $x_1$   $y_1$

so lets say that the general equation of such a circle is and the center is  $g$   $f$

so obviously if we join the center and this point  $x_1$   $y_1$  then this angle is  $90^\circ$  degrees because since this circle touches the straight line at  $x_1$   $y_1$  this straight line is actually a tangent to the circle at this point

so therefore the slope of the product of the slope of this straight line which is the perpendicular to the tangent from the centre and the slope  $m$  of the straight line itself

so the product of slope  $m$  and the slope of this perpendicular should be minus one

so that is how we are going to start the slope of this perpendicular is clearly  $y_1 - f$  which is  $y_1 + f$  divided by  $x_1 - g$  which is  $x_1 + g$  times the slope of the straight line which is  $m$  should be minus one and therefore from here what we end up getting is that  $m$  times  $y_1 + f + x_1 + g$  equals zero

so now what we can do is we can use this information over here to derive the equation of the family of all such circles

so from here what we see is that we can derive an equation for  $g$

so what we see is that  $g$  is equal to  $-x_1 - m(y_1 + f)$

so we will what we will do now is we will take this expression for  $g$  and we will substitute it over here and let us see what we get then

so when we do that what we get is  $x^2 + y^2 + 2mx + 2ny + c = 0$

so instead of  $g$  we use the derived expression which is this plus two times  $f$  times  $y$  plus  $c$  equals zero but we also know that the radius of since the center is  $g$   $f$  the radius or the square radius of this circle is from this equation this squared radius is  $g^2 + f^2 - c$  and just by looking at ah this figure this square radius is this squared equilibrium distance between this point and the centre which is  $(x_1 - g)^2 + (y_1 - f)^2$

so therefore this and this should obviously be equal

so from here we see that  $c$  must be equal to  $g^2 + f^2 - (x_1 - g)^2 - (y_1 - f)^2$

so this is another expression that we get for  $c$  in terms of the coordinates of the center of the circle and of course the fixed point  $x_1$   $y_1$

so we will use this right hand side over here then what we get is  $x^2 + y^2 + 2mx + 2ny + c = 0$  but instead of  $c$  we are going to use the expression which



get the equation of the family of circles to be of this type where  $k$  is sum which is the free parameter

so even if we go back to the general equation of family of circles which touches the straight line  $y - y_1 = m(x - x_1)$  at the point  $(x_1, y_1)$

so this is what we had derived

so this can be further simplified and i leave that as an exercise

so this entire equation can be rewritten as  $(x - x_1)^2 + (y - y_1)^2 + k = m^2(x - x_1)^2$  equals zero and its not very difficult i think what we just need to do is we need to replace this  $x^2$  and  $y^2$  by introducing these two terms

so we can write this  $x^2$  as  $(x - x_1)^2 + 2x(x_1 - x_1) + x_1^2$  and we can write  $y^2$  as  $(y - y_1)^2 + 2y(y_1 - y_1) + y_1^2$  and then we can write the remaining terms as it is

so this term here is  $-2x(x_1 - x_1)$  the next term is  $-2m(x - x_1)(y - y_1)$

so this and this will get cancelled then here we have  $+2f(y - y_1) - 2mf(x - x_1) + (x - x_1)^2 - (y - y_1)^2 + 2(x - x_1)^2 + 2m(x - x_1)(y - y_1) - 2f(y - y_1) = 0$

so what we ah see is that this  $-2(x - x_1)^2$  and the  $-2(x - x_1)^2$  here they become  $-4(x - x_1)^2$  which gets cancelled with the  $+4(x - x_1)^2$  here and then the remaining terms can be simplified further we get

$+2(y - y_1)^2$  and this can be combined and written as  $-2(y - y_1)^2$  and then this entire thing can be written as  $2(y - y_1)^2 - 2m^2(x - x_1)^2 = 0$  further the term  $2m^2(x - x_1)(y - y_1)$  and  $-2m^2(x - x_1)(y - y_1)$  can be combined and we get  $+2(y - y_1)^2 = m^2(x - x_1)^2$

so we have combined this and this term and then we can also combine this and this term we get  $+2f(y - y_1)$  and then what remains is just this term with ah this term here which is going to give us  $+2mf - 2mf = 0$

so if we again simplify this further we get  $+2(y - y_1)^2 = m^2(x - x_1)^2$  times  $(x - x_1)^2$  which is combining this and this term and then  $+2f(y - y_1) - m^2(x - x_1)^2 = 0$  and then of course this and this is same

so we can write  $+2(y - y_1)^2 + f = 0$  and this is actually the parameter  $k$

so finally we do get this form for the family of circles when it is clear that irrespective of the value of  $k$  if you put  $x = x_1$  and  $y = y_1$  here the left hand side evaluates to zero which shows that the point  $(x_1, y_1)$  lies on this circle on all these circles whatever may be the value of  $k$

so that finishes our discussion on the equations of family of circles next we are going to take up the how to derive the equation of a chord if we are given the midpoint of a chord suppose that lets say that we have a circle here having center at  $(-g, -f)$  let us say there is a chord and let us say that we know that the midpoint of this chord is given to us and it is  $(x_1, y_1)$  and then we are asked to find the equation of the heart

so clearly we know that if we join the midpoint of the heart to the center of the circle then this angle is  $90^\circ$  let us say there is any other point  $(x, y)$  on the chord then the slope of this chord is on the slope of this straight line is equal to  $\frac{y_1 - (-f)}{x_1 - (-g)}$  which is this but since these two straight lines are at ninety degrees the product of this slope should be minus one and therefore this times this should be minus one which can be simplified to and this is the equation of the chord

so next we are interested in finding the equation of the cord

so let us say that

so next let us say that we are given a point  $p$  having coordinates  $x_1, y_1$  and a circle whose equation is also given to us let us say that this point  $p$  is outside this circle

so clearly there are two tangents  $pt_1, pt_2$  from the point  $p$  to the given circle and if we join  $t_1, t_2$  it is nothing but a chord

so the objective now is to find the equation of this chord of contact

so this is called the chord of contact

so this equation of the chord of contact  $t_1, t_2$  for a given point  $p$  outside a given circle having this equation

so of course we know the radius of this circle  $r$  which is given by square root of  $g^2 + f^2 - c$  the length of these tangents can also be found out because we know that this is  $90^\circ$  and if we join  $p$  with the center  $o$  then  $pt_1, o$  is a right angle triangle therefore from the pythagoras theorem we know that the square distance  $po$  is equal to  $l^2 + r^2$  further  $po^2$  is since we know these two coordinates  $po^2$  is and therefore if we substitute this expression over here we get  $l$  is equal to square root of  $po^2 - r^2$

so we know  $l$  also now consider ah the circle let us consider the circle whose center is at  $p$  and having radius equal to  $l$

so this circle shown in red has center  $p$  radius  $l$  clearly this circle or it passes through  $t_1$  and  $t_2$  and therefore it is clear that  $t_1$  and  $t_2$  are the point of intersection between this red circle and the given circle whose equation is this and therefore the equation of this chord  $t_1, t_2$  which we want to find out is nothing but the equation of the radical axis between the red circle and the given black circle and this we know earlier

so the equation of this red circle is  $(x - x_1)^2 + (y - y_1)^2 = l^2$  and that of the given circle is this equation

so let us say this is  $S_1$  this is  $S_2$

so when we write it we can we have to write it like this and the equation of the radical axis will be simply  $S_1 - S_2 = 0$  therefore the equation for the horde of contact is given by we just need to subtract  $S_2$  from  $S_1$  and we get on further simplification we get that is

so this is the equation of the horde of contact

so next we will also for the same situation

so we know how to find  $l, r$

so lets say we are asked to find the length of this cord of contact

so it is not very difficult because we see that this angle will be  $90^\circ$  let's say if this angle is  $\theta$  then this angle is  $90^\circ - \theta$  but because this entire angle is  $90^\circ$  this angle is also  $\theta$  further this point which is the intersection of the chord of contact with the straight line joining the center  $o$  to the point  $p$  this point is the midpoint of this chord and therefore if this length is  $x$  then this is also  $x$

so let this midpoint be  $m$  and let us say that this small length here  $om$  we will denote it as by  $h$

so we have this triangle  $t_1, m, o$  like this and we also have the triangle  $p, t_1, o$  one can see that all the three angles of these two triangles are same and therefore triangle  $t_1, m, o$  is similar to triangle  $t_1, p, o$  and therefore the ratio of the corresponding sides must be the same hence  $x$  by  $l$  is equal to  $h$  by  $r$  is equal to  $r$  by square root of  $r^2 + l^2$  because of similarity and from here we get  $x$  equal to  $rl$  divided by square root of  $r^2 + l^2$  and  $h$  equal to  $r^2$  by square root of  $r^2 + l^2$  and  $h$  equal to  $r^2$  by square root of  $r^2 + l^2$

square and therefore the length of the chord of contact  $t_1 t_2$  is simply twice that of  $x$

so which is 2 times  $x$  which is  $2rl$  by square root of  $r^2 + l^2$   
so we can find many other interesting things about this given this situation for example we can find out the area of the triangle  $pt_1 t_2$

so  $pt_1 t_2$

so how do we do that now clearly the total area of the quadrilateral  $pt_1 t_2 o$  is the sum of the areas of these two

so this triangle area plus area of this triangle will come out to be  $l$  into  $r$  because the area of  $pt_1 o$  is half into  $l$  into  $r$  and the area of  $pt_2 o$  is the same further the area of  $ot_1 t_2$  is simply  $h$  into  $x$  we can use the previously derived values of  $h$  and  $x$  and we get this to be  $r^2 l$  by  $r^2 + l^2$  square and therefore area of  $pt_1 t_2$  is equal to area of  $pt_1 o t_2$  minus area of  $ot_1 t_2$  which is equal to which equals this expression

so we can also find out this angle subtended by the two tangents from this point  $p$

so clearly this angle is simply  $\tan^{-1}$  of  $r$  by  $l$  therefore this entire angle is  $2 \tan^{-1}$  of  $r$  by  $l$  which is simply

so we can use a  $\tan^{-1} a + \tan^{-1} b$  formula and this is what we get finally another type of question that can be asked is for the same setup we could be asked to find the equation of the circle which circumscribes the triangle  $pt_1 t_2$

so we have to find the equation of this circle in red which passes through  $p$   $t_1$  and  $t_2$

so this circle passes through  $t_1$  and  $t_2$  and we know that  $t_1$  and  $t_2$  are the point of intersection of the given circle in black and this circle in blue

so this circle in blue has its center at  $p$  and the radius equal to  $l$

so clearly this circle in blue passes through  $t_1$  and  $t_2$  and it intersects the given circle in black at  $t_1$  and  $t_2$

so therefore if we consider the family of all circles which pass through the points  $t_1$  and  $t_2$  which are point of intersection of the blue and the black circle then this red circle should belong to that family of circles

so that is how we will try to find out the equation of this red circle

so the equation of this given circle is  $S = 0$  and that of this red circle is simply which can be simplified to

so these two circles are known to us the family the equation of family of all circles which pass through the point of intersection of these two circles is the  $\lambda$  is a free parameter

so we change  $\lambda$  by changing  $\lambda$  we will get different circles which pass through  $t_1$  and  $t_2$  this is the point of intersection of these two circles

so this equation is now one such circle is this red circle but we know that the red circle also passes through  $x_1$   $y_1$  therefore if you put  $x$  equal to  $x_1$  when  $y$  equal to  $y_1$  then this left hand side should evaluate to zero

so when we do that we get which implies that  $\lambda$  must be equal to  $1/l^2$

so therefore we have found out the value of  $\lambda$  and then what we need to simply do is we need to because  $x_1$   $y_1$   $g$   $f$  and  $c$  are known to us they are given to us  $l$  is also given known to us in fact in fact  $l^2$  is we use this formula and  $r^2$  is the square of this if we use that here what we will see is that this numerator is  $l^2$  and therefore  $\lambda$  equal to one hence the equation of this red circle is nothing but we just need to put  $\lambda$  equal to one in this equation and therefore this is the equation of the red circle thank you