welcome to the 12 lecture on circles

so in this lecture we will start a new topic called family of circles which is similar to the topic of family of straight lines

so here we will basically talk of writing general equations for all circles which satisfy a common property for example we can write the equation of all possible circles which pass through any two given points or for example the equation of all circles which pass through the intersection of two given circles

so that is what is going to be the topic of this lecture

so let us start with the with the first scenario where lets say that we have been given two circles equations whose equations are s one equal to zero and s two equal to zero

so s one is this

so s one is this second degree polynomial in x and y when s two is this other second degree polynomial in x and y

so this represents the first circle s one equal to zero and this represents the second circle s two equal to zero and let us say that these two circles intersect each other at two points p and q

so we are now interested in finding the equation of all those circles which pass through these two points of intersection for example one such circle could be this circle which i am drawing now another circle could be something like this yet another circuit will be something like this as you will soon realize there are infinitely many circles which can be drawn which will pass through both these points of intersection of these two given circles but the objective here is to find a general equation or an equation which will have some parameter such that if we change that parameter we should be able to get the equation of any such circle which passes through these two points to do that we will use the concept of radical axis and let us do a little bit of recollection on that

so if we recollect for any two intersecting circles if we recollect for any two intersecting circles the radical axis was given by the unique straight line which joins the two points of intersection

so in this case of s one equal to zero and s two equal to zero the radical axis would be this red straight line which passes through both p and q

so this would be the radical axis of s one and s two but then suppose that we have a general circle s whose equation is given by x square plus y square plus two g x plus two f y plus c equal to zero

so we are asked to find the general equation of such circuits which pass through both these points now the equation of this straight line or which is the radical axis of these two circles was simply given by s one minus s two equal to zero

so if we subtract s one and s two and equate to two zero what we will get is two into g 1 minus g 2 into x plus 2 into f 1 minus f 2 into y plus c 1 minus c 2 equals 0.

so this is the equation of the radical axis between these two circles now clearly if there is any other general circle given by this equation which also passes through these two points then it is clear that s and any such general circle s and the given circle s one will also intersect at p and q because p and q lies on s one and we are considering all those circles s which pass through p and q

so p and q must also lie on s and hence p and q the points p and q must be common to both s and s one and therefore s and s one must intersect at p and q and therefore the radical axis between the circles s equal to zero and s one equal to zero is given by the straight line equation s minus s one equal to zero

so the radical axis between these two circles will be between any circle which

also passes through p q and this circle s one will be s minus s one equal to zero but this radical axis is also a straight line which passes through p and q and the equation of this straight line will be s minus s one

so this minus this

so the equation of this ah will be s minus s one will will be two into g minus g one x plus two into f minus f one y plus c minus c one equals zero but then it is noted that the points p and q lie on this straight line as well as this straight line also between any two points p and q there is only one unique straight line and hence this equation and this equation must be representing the same straight line

so what we have

so far is that 2 g minus g 1 into x plus 2 f minus f one into y plus c minus c one equal to zero

so this equation which is the radical axis between s and s one and this other straight line equation which is the radical axis between s one and s two

so these two are nothing but the same straight line what that basically implies is that if we take this first equation and if we multiply it with some real valued real number q

so since they are they are the same equation there must exist some real number q such that if we multiply this first equation by that q we must exactly get the second equation because they are nothing but the same straight line and therefore what must hold is that after we multiplied this first equation by q we get two q g minus g one into x plus two q f minus f one into y plus q into c minus c one equals zero

so after multiplying with q we should exactly get this equation which basically means that this and this should be exactly the same equation coefficient by coefficient and for that to happen it must be true that g one minus g two must be equal to q into g minus g one f one minus f two must be equal to q into f minus f one and c one minus c two must be equal to q into c minus c one remember here g one g 2 f 1 f 2 and c 1 c 2 are all known now in order to find the general equation there must be some relation between g f and c for this general equation

so this because this is representing the equation of all those circles which pass through both these points

so let us see what is this what properties do g f and c satisfy

so from here what we can see is that g will be equal to g one minus g two by q plus g one similarly f will be equal to f one minus f two by q plus f one and c will be equal to c one minus u two by q plus c one

so this is what we get this can be simplified

so this is what we get

so now let us try to if we go back to this general equation we replace g f and c by this right hand sides and if we do that we should be able to get the general equation where on the right hand side will have q

so this in this equation will have this parameter q and by changing that parameter q will get different circles from the family of circles

so when we do that what we get is x square plus y square plus two x into g

so instead of g we put this right hand side plus two y into f

so instead of f we put this right hand side plus c

so instead of c we put this expression equal 0 and then let us what we can do now is that we can write 1 plus x square as equal to 1 plus q by q into x square minus 1 by q into x square

so this x square will replace if this with these two different terms and the same thing will do with y square and if we do that we will get s to be one plus q by q into x square plus y square plus two g one x plus two f one y plus c one

minus one by q times x square plus y square plus two g two x plus two f two y plus c two equals zero now clearly if we go back to this slide it is clear that q cannot be zero because if q were to be zero then this and this will not match because clearly this is not this is a non zero equation then this is also a non zero equation whereby what i mean is that ah in this equation g one and g two are not equal f one and f two

so out of g one g two f one and f two c one and c two it is clear that g one g1 and g2 the at least one of these two is non-zero at least

so either g1 minus g2 is non-zero or f1 minus f2 is non-zero both cannot be zero because if both are zero then the centers are the same then the centers are the same in which case what we have is concentric circles and concentric circles will not intersect each other if we if we go back to the first slide we said that we are talking of two circles which intersect each other

so clearly here there is at least one of these coefficients which is non zero and therefore this is not a trivial equation and hence q will never be zero because if you multiply this by q will get a zero equation and then then there is no way that this and this can be the same equation because if you multiply this by q equal to zero if q were to be zero then when we multiplied with this equation we get a zero equation whereas this equation is not a zero equation

so this real valued q is not zero and therefore what we can do is that we can multiply this entire equation with q

so when we do that we get rid of the denominator here

so what we then get is that one plus q into s one minus s two is equal to zero

so this is the general form of the equation that we get that we get and we can also write it write this equation as one plus q into s one minus one by one plus q into s two equal to zero and we can do this ah because and further what we also see is that we can write this as plus equal zero but this is nothing but this equation where k is minus one by one plus q and since q is real k is also real valued since q is not equal to zero k is not equal to minus one

so therefore this is the general equation of all those circles which pass through the point of intersection of the two circles s one equal to zero and s two equal to zero but this k should be not equal to minus one can be any real value not equal to minus one let us take a little example to illustrate this

so let us say that we have two circles x square plus y square plus two x plus four y minus four equals zero this has center minus one power minus two and has a radius of three that the other circle will be s two equal to x square plus y square plus six y equals zero this has center zero comma minus three and radius equal to three as we see clearly the distance between the two centers is equal to square root of two which is less than the sum of the radius because the sum of the radius is six and this is clearly greater than the because the absolute difference between the radius is zero

so this is the situation which basically means that the two circles are intersecting at two points and now we would like to find out the equation of all possible circles all possible circles which pass through the two points of intersection of these two circles

so the general equation of all those circles will be s 1 plus k times s 2 equal to 0 where k is real and k is not equal to minus one

so for this example will have s equal to

so s one is x square plus y square plus two x plus four y minus four plus k times s two is x square plus y square plus six y equals zero clearly if we put k equal to zero then we have s to be simply equal to s one and if we choose as k tends to infinity this equation will correspond to s two equal to zero in the limit

so this can be further refined you know this can be further written as 1 plus k

x square plus 1 plus k y square plus 2 x plus 4 plus 6 k into y minus four equals zero

so this is the general equation of all such circles we just have to keep changing the value of k and will get a different circle each time but we must make sure that k is not equal to minus one because if k is equal to minus 1 then the coefficient of s square and y square are 0 and therefore with k equal to minus 1 what we are going to get is simply s 1 minus s 2 equal to 0 which is nothing but the equation of a straight line which is the radical axis of s one and s two and that will not be the circle the equation of a circle that is why we have said that k must not be equal to minus one

so that was the first type of family of circles another type of family of circles is let us say if we are given a circle s prime equal to zero

so let that be given by this equation and let us say we have a straight line l which is given by this equation

so these two are given to us and let us say that these two this straight line and this circle intersect at two points or they could even be touching at just one point and then we are interested in the equation of all those circles which pass through this point of intersection which pass through this point of intersection of this given circle s prime and this straight line the given straight line l now it is clear that if we have any other circle

so let us say we have any other circle which also passes through these two points of intersection then it is clear that the radical axis

so let this be the general circle s equal to zero then it is clear that the radical axis between this circle s and the given circle s prime must be this straight line

so the radical axis between s equal to zero and s prime equal to zero must be l equal to zero this is because the straight line l joins these two points of intersection of

so this the straight line intersects these given circle s prime at these two points p and q now if we have any other circle s equal to zero which also passes through these two points then it is clear that the straight line joining these two points has to be the radical axis between s equal to zero and s prime equal to zero but since there is always only a unique line joining two points this straight line was nothing but b the equation of this straight line must be nothing but the this equation the radical axis between s equal to zero and s prime equal to zero is given by s minus s prime equal to zero that is

so let us say our general circle has this equation then s minus s prime will give this equation but then this equation must be nothing but this equation of the straight line l which is since these two equations must be the same there must exist a q not equal to zero such that if we multiply this straight line equation by this q we must exactly get the equation for s minus s prime which is this because we have argued that these two must represent the same straight line

so when you multiply this by q we get and now we can equate term by term of basically coefficient by coefficient because this equation and this equation are identical and therefore q must be such that m q must be equal to two times g minus g prime n q must be equal to two times f minus f prime and p q must be equal to c minus c prime from these three equations from this equation we get two g equal to two g prime plus mq from here we get two f equal to two f prime plus n cube and from here we get c equal to c prime plus p q now going back to the general equation of the family of circles we substitute these expressions for two g two f and c in this equation and we get s equal to x square plus y square plus instead of two g we write two g prime plus m q into x plus 2 f prime plus n q into y plus c prime plus p q equals 0.

and then we if we separate out the terms we can write this expression as x

square plus y square plus two g prime x plus two f prime y plus c prime plus q into m x plus n y plus p equal to zero but note that this is nothing but s prime this is the polynomial s prime and this is the first degree polynomial l and therefore the general equation for us all such circles becomes s prime plus q l equals zero

so s is equal to s prime plus q l equals zero

so as we change this q

so q is real valued here as we change the skew we get different circles from this family of circles which will pass through these two points p and q where p and q were the points of intersection of the given circle s prime with the given straight line l another family of circles can be computed suppose if we have two given points x one y one and x two y two and it is said that we would like to find out the equation of all those circles or the general equation of all those circles which pass through these two points

so as long as these two points are not the same there are infinitely many circles which will pass through both these two points how do we find the general equation of all these circles

so to do that what we can do is we can consider the following circle

so we join these two points and we find the midpoint of this line segment when the coordinates of the midpoint will be x one plus x two by two n y one plus y two by two and we construct a circle with radius equal to half the length of this line segment clearly that circle will touch or will go through both these two points

so let the points be p and q

so from p and q we get another we get the equation of this circle

so let will denote this we can easily compute the equation of the circle and will denote this circle by let us say s prime

so this circle will simply be given by the equation x minus center holes x coordinate of the center x minus x coordinate of the centre whole square plus y minus y coordinate of the centre the whole square must be equal to the square radius now square radius is one fourth of the squared diameter and the square diameter is nothing but the square distance between the points p and q which is given by x one minus x two whole square plus y one minus y two whole square

so this is simply the

so if we bring this term on to this side will have a minus here and then equal to zero

so this square plus this square minus this thing equal to zero

so this is the equation of this circle clearly this this the moment we define the moment we are given these two points s prime is automatically defined and we also see that if we see this let us extend this straight line joining p and q

so the equation of this line let us say will represent it with l will say that the equation of the straight line is l equal to zero and that straight line equation is also easy to find out because ah

so this straight line equation will be given by y minus y one by x minus x one is equal to y one minus y 2 by x 1 minus x 2 and which can be simplified to x minus x 1 into y one minus y two plus y minus y one into x two minus x one equals zero

so let us say this is our l

so l

so l is equal to this single degree polynomial in x and y and that is equal to zero

so now what we have is we have a circle s prime equal to zero we have a straight line l equal to zero and we know that this straight line and this circle intersect at these two points

so essentially what we have here is that we are trying to find the equation of all those circles the general equation of all the circles which pass through the point of intersection of this circle s prime equal to zero and this straight line l equal to zero because this circle s prime equal to zero by design will intersect this straight line at exactly these two points p and q that were given to us this is because of the manner in which we have constructed this circle s prime and since we are trying to find the general equation of all those circuits which pass through p and q it is the same as finding the general equation of all those circles which pass through the point of intersection of s prime equal to zero and the straight line l equal to zero and this is something that we have already done in just in the previous slide where we had said that the general equation for all circles is given by s prime plus k l equal to zero where k is real value

 so this is how we can get the

 so now we just have to replace this s prime with this this left hand side this equation

 so this will basically become x minus x one plus x two by two whole square plus y minus y one plus y two by two whole square minus x one minus x two whole square plus y one minus y two whole square by four

 so this thing plus k times this expression

 so k times this thing plus this ah this polynomial for s prime

 so the polynomial for s prime is this one plus k times the single degree polynomial for the straight line l

 so this plus k times this must equal to must be equal to zero

 so this is the equation of the all such circles which will pass through these two given points x one y one and x two y two and k is real valued

 so we can keep changing k and we can keep on getting different different circles let us take this example to illustrate the what we just discussed

 so let us say we have two points p which is two comma 0 and another point q which is 6 comma minus 4 and we would like to find the general equation of all those circles pass through these two points this this is the straight line joining these two points the midpoint is this point o whose coordinates are four comma minus two clearly this the distance o p is equal to square root of eight and therefore if we draw a circle with center as o the radius is square root of eight

 so that circle will be something like this and clearly that these two points will lie on that circle because we have chosen the radius to be equal to root eight which is half of this length and we have chosen the center of this line segment to be the the midpoint of this line segment to be the center of that circle

 so clearly p and q will lie at diametrically opposite ends of this on this circle

 so p q will be a diameter and the equation of this circle s prime will be given by x minus four whole square plus y plus two whole square is the square of the radius which is eight

 so this is the equation of the this circle which is can be simplified to sorry

 so this is the equation of this circle but this can be simplified to s written as it can be written as s prime equal to x minus four whole square plus y plus two whole square minus eight equals zero that is x square plus y square minus eight y plus x square plus y square minus eight x plus four y plus twelve equals zero

 so this is the equation of the this circle similarly it is not very difficult to write them down the equation of this straight line joining p and q the equation of this straight line joining the two points p and q will be given by y

minus zero divided by x minus two equals which is minus one and therefore this equation of this straight line is x plus y minus two equals zero

so this straight line will be given by l equal to x plus y minus two equals to zero and now since we know that this circle s prime and this straight line l by design they will intersect at these two points

so we are essentially trying to find out the family or the the equation of all those circles of the family of circles which pass through the point of intersection of this circle s prime equal to zero and the straight line l equal to zero which is given by this general equation s equal to s prime plus k l equal to zero which if we substitute the polynomials for s prime and l over here we get x square plus y square plus k minus eight x plus k plus four y plus twelve minus two k equals zero

so this is the general equation of the of all such circles that will pass through both these points and one can check it also

so so this is the general equation that we have found out for all those circles which will pass through the points p is two comma zero and q six comma minus four

so clearly this is an equation of a circle because there is no coefficient of x times y the coefficient of x y is zero and the coefficient of s square and y square are the same further g square plus f square minus c is k minus eight whole square by four plus k plus four whole square by four minus twelve minus two k and which comes out to two k square minus 8 k plus 80 minus 48 plus eight k divided by four which is equal to two k square plus thirty two by four which is strictly greater than zero

so clearly this has to be the equation of some circle now let us also see whether these two points indeed lie on this circle

so if we replace to check that this point two comma zero lies on the circle will put x equal to two y equal to zero on the left hand side and see whether this polynomial equation with x equal to two and y equal to zero whether it evaluates to zero or not

so the value of this polynomial at the value of this polynomial with x equal to two and y equal to 0 will be 4 plus 0 plus k minus 8 times 2 plus 12 minus 2 k which will be equal to 4 plus 2 k minus 16 plus 12 minus 2 k

so this cancels and 4 plus twelve is sixteen minus sixteen is zero

so indeed the left hand side ah basically this polynomial evaluates to zero which means that this point does lie on this circle s equal to zero which means that or no matter what value of k we choose no matter what value of k we choose we get some circle and that circle will pass through this point two comma zero and the similar thing we will get if we test ah this equation with for this point six comma minus four

so with that we will end this lecture in the next lecture we will also take one more remaining case where we will try to find the family of circles and then we will try to solve some challenging problems on family of circles from previous exams thank you you