welcome to lecture 11 of circles
 so in this lecture we will start with ah deriving the condition for two circles
to intersect each other
 so we will derive that condition with more rigor ah if we recollect in the
previous lecture we had stated that if the distance between the centers of the
two circles is less than the sum of the radius and if this distance between the
two centers is also greater than the absolute difference of the radius then
under this condition we had said that the two circles will intersect at two
points although we had not shown this rigorously
 so moving ahead
 so if we recollect we had said that if there are two circles s one equal to
zero and s two equal to zero and let us say that the radius of this circle s one
equal to zero is r one and the center is this point o one and let us say that
this circle given by equation s two equal to zero has radius r two and center at
o two then we had made this remark that if the distance between the two centers
is less than the sum of the radius and if it is greater than the absolute
difference between the radius of the two circles
 so when this condition happens then we said that the two circles will intersect
at exactly two points and we also said we also said that if
 so this is
 so if this condition happens then the two circles intersect at two points we
also said that if the distance between the two circles is equal to the sum of
the radius then the two circles touch each other externally at exactly one point
we also said that if this distance between the centers of two circles is equal
to the absolute value of the difference of the radius then in that case the two
circles touch each other internally
 so we had also explained this ah through example
 so we have drawn circles and shown ah how these conditions can arise but we had
not proven these statements formally or rigorously
 so we will try to do that now
 so let us say that these are the two circles
 so we have this this is the equation of the circle first circle given by s one
equal to zero having center o one and lets say that this is the other circle s
two equal to zero
 so in subsequent discussion we will assume that without loss of generality we
will assume that the radius of the first circle is greater than the radius of
the second circle or it can even be equal to the radius of the second circle let
us denote this centre of the second circle by o2 and in that case then this line
segment has length d o1o2 this is one of the points where these two circles
touch will denoted by p and will connect o one and o two to this p now let us
try to examine what happens if or let us say under which conditions will it
 so happen that these two circles will touch each other at exactly one point
 so if we also again recollect our previous lectures then let us say that the
the coordinates of this point one which is the center of the first circle are a
b and lets say that the coordinates of the center of the second circle are c
comma d then the length of one p is obviously r one length of o two p is r two
and lets say that the coordinates of this point p are denoted by x and y and
then we and also said that let us say that
 so let us say that this green dotted line is parallel to the x axis and
 so is this green dotted line also and let us say that this angle of one p with
respect to the x axis
 so this angle is theta similarly the angle of o to p o two p with respect to
the green dotted line let us say that it is denoted by phi we also drop a
perpendicular from o two on to this green dotted line

so this perpendicular let us denote this point by m where the perpendicular meets the green dotted line it is clear that if we if we we can write this point x the coordinates of this point p can be written in polar form with respect to both the circles

so when we write this point p the coordinates of this point p in polar form with respect to the first circle we see that x is equal to a plus r one cos theta and y is equal to b plus r one sin theta similarly x and y can be expressed in terms of polar form with respect to the second circle in which case x is c plus r two cos phi y is d plus r two sin phi now we we can equate ah this and this equation and this and this equation whereby we get that a plus r one cos theta is equal to c plus r two cos phi and b plus r one sin theta is equal to d plus r two sin phi now because we have assumed r one to be greater than equal to r two what we will do is that we will will take this a on to this side

so we get r one cos theta c minus a plus r two cos phi and r one sin theta is d minus b plus r two sin phi and now we can square this equation in this equation and add them up

so when we square these two equations and add them up we get r one square cos square theta plus square of this left hand side r one square sin square theta equals the square of this plus the square of this which is c minus to the square of this thing this term is c minus a square plus r two square cos square phi plus two c minus a r two cos phi and then the square of this particular term here is d minus b whole square plus r two square sine square phi plus two into d minus b into r two sin phi and if we were to simplify that using the fact that cos square theta plus sin square theta is one for any theta and similarly cos square phi plus sin square phi is also one we use that fact then we get r one square is c minus a square plus d minus b square plus r two square plus two r two into c minus a cos phi plus d minus b sin phi now c minus a the whole square plus d minus b whole square

so c c minus a whole square plus d minus b whole square is nothing but the squared length of this of the distance between the two centers which

so this we can denote by d square d o one o two square and let us say that here we for this term we multiply and divide by d over no two and there by we get we can put a d o one o two here and then we put a d o one o two in these two terms eventually we get r one square is r two square plus d o one o two square plus two r two d o one o two times c minus a by d one o two into cos phi plus d minus b by d one o two into sin phi

so let us see what are these two terms you know these uh c minus a by d one o two and d minus b by d one o two if we go back to the previous figure what we see is that if we focus on this try right angle triangle o one o two m then we see that o one m is nothing but c minus a o two m is nothing but d minus b and let us denote let us denote this angle m o1 o2

so which is this angle let us denote it by alpha

so it is easily seen that c minus a divided by d over one o two is nothing but cos alpha and d minus b divided by d o one o two is sine alpha

so if we use that we get this to be cos alpha and this to be sine alpha and therefore we have this expression but this is of the form cos a cos b plus sin a sin b and this is nothing but cos of phi minus alpha

so the it is also clear that the the value of alpha depends on nothing but the coordinates of the centers of the two circles which is given to us

so alpha is known to us therefore what is unknown to us is the value of phi and theta because we do not know we do not exactly know the points of intersection of the two circles and that is what we our effort is to devise a method our effort here is to come up with a method to find the to characterize the points of intersection

so the points of intersection were given by these two polar forms
so the moment we are able to ah find phi it is clear from either phi or theta
so we have emulated theta here
so now ah what we have here if we see this equation we basically have a trigonometric equation in terms of phi
so we can solve
so the this equation is precisely this equation r one square is r two square plus d one o two square plus two r 2 d o 1 o 2 into cos of phi minus alpha
so everything in this equation is known to us because the we know the equation or two circles
so we know the radius we know the distance between the centers we know this angle alpha now what is not known is phi which we can know by solving this equation once we know phi we can very easily put the value of phi here and we can get the coordinates of this point of intersection
so from this equation
so from the previous slide we have cos of phi minus alpha is r one square minus r two square plus d one o two whole square upon two r two d o one o two
so essentially we have to solve this equation to find the value of phi which can be done graphically like this
so we will plot cos of phi minus alpha on the vertical axis versus phi on the horizontal axis and the graph of cos of phi minus alpha might look something like this
so let us say this is alpha
so when phi is zero cos of phi minus alpha alpha cos of i minus alpha is cos of alpha which is let us say this value when phi is alpha cos of phi minus alpha will attain its maximum value of one
so let us say we have something like this
so this is one complete cycle
so this value would be two pi and this value this value of phi where it attains its minimum
so we see that when if we we have phi equal to alpha plus pi then cos of phi minus alpha is minus one
so which is the minimum possible value it can attain and it will attain that at phi equal to alpha plus pi which is this point now if it is clear that if this value on the right hand side of this equation has an absolute value which is less than equal to one then we will have a solution
so we can represent this value by a line by a horizontal displacement from the x axis
so let us use a green dotted line too
so suppose if this value is let us say less than one suppose let us say it is equal to half
so if this value is is equal to half
so half is somewhere here because this is one
so this is half and then corresponding to this value which is let us say half we will draw a dotted line which is parallel to the x axis and is displaced from the x axis by half
so the places where this green dotted line is going to cut the curve for cos phi because this green line is the is geometrically the equation of the line y equal to this value for this graph
so obviously you know wherever this green line is going to cut the curve for cos phi minus alpha these two values will be equal and therefore in this case if this were to be half then the two values of five would be this value and this value in fact it is easily seen that as long as this value on the right hand side has a modulus value strictly less than one if the modulus value is strictly

less than one then one can easily see that there will always be two solutions of phi because of this complete one complete cycle here there will be two different values of phi which will satisfy this equation and that will happen if this right hand side has an absolute value which is strictly less than one for example we took half and then we saw ah this and this were two different values of five however if this

 so so that particular case where this right hand side has an absolute value less than one will correspond to the scenario where the two circles intersect each other at two points at exactly two points however we see that if this right hand side were to be equal to one either one or minus one

 so if this right hand side is equal to one then the green dotted line is something like this in which case the green dotted line corresponding to this value of one touches this curve at only one place exactly one place which corresponds to phi equal to alpha which corresponds to phi equal to alpha and therefore the two circles will touch each other only at one place which whose coordinates are given by those coordinates will be given by this equation but with phi equal to alpha if this right hand side where if the radius and the distance between the centers were such that this right hand side is equal to one similarly when it is equal when this right hand side is equal to y minus one even then when it is equal to minus one then the green dotted line is something like this and even in that case there is exactly one value of phi which satisfy this equation which basically means that there is only one point x y corresponding to phi equal to alpha plus pi where these two circles will touch each other

 so in this special case where this right hand side is equal to one or minus one we see that the two circles will intersect each other at only one point which is basically which basically means that they will touch each other at one point

 so next we will try to see what which how will try to characterize these two cases where the circles touch each other in terms of this ah of r 1 r 2 and the distance between the two centers

 so we just now had said that

 so let us say that r one square minus r two square plus d o one o two whole square by two r two d one comma o two is equal to lets say minus one and let us see what this corresponds to

 so which is equal to cos of phi minus alpha

 so this will happen if and only if

 so if we try to ah just solve this what we see is that r one square is equal to r two square plus d o one o two whole square minus two r two d o one o two which means that r one square is equal to r two minus d o one o two whole square and this implies that r one is either equal to r two minus d o one o two or r one is equal to d o one o two minus r two now clearly this case is not possible because we had initially said that r one is greater than equal to r two and d the distance is always a non negative quantity the distance between the two centers and therefore this is this possibility is this is not possible

 so the only possibility is this and this this condition is nothing but that the distance between the centers is equal to the sum of the radius

 so this is one condition in which the right hand side is equal to minus one but let us try to also geometrically see ah what does this mean

 so when the distance between the two centers is equal to the sum of the radius this expression is minus one which means that cos of phi minus alpha is minus one or which basically implies that phi is equal to

 so phi minus alpha is basically pi or basically that phi is equal to alpha plus pi

 so what we have is that phi is equal to alpha plus pi now we go back to this

figure let us see what it means when we say that phi

so we have said now that phi is equal to alpha plus pi

so we want to investigate this ah situation

so ah let us see this angle

so this angle here is

so this is 90 degrees this angle here is obviously pi by 2 minus alpha and therefore this angle o 1 o 2 p o 1 o 2 p this angle which let us denote by beta

so this angle beta can be calculated because this is phi this is 90 this is pi by 2 minus alpha

so it will come out to be

so beta will come out to be pi beta equal to pi plus alpha minus phi now when phi is equal to alpha plus pi which is the scenario we are considering right now then if we replace this phi by alpha plus pi we see that if this happens then beta is actually equal to zero beta is equal to zero but what does beta equal to zero mean

so so let us now focus on this triangle o one o two p

so beta equal to zero means that

so this angle beta is going to collapse to zero which means that this point p is on this is this point p must lie on this somewhere in between the points one o two on the straight line of one o two

so basically this triangle o one p o two

so basically this triangle o one p o two becomes a straight line with the point p somewhere in between the points one and o two

so this is what will happen when beta collapses to zero when beta is equal to zero and in that case what this basically means is that

so what it basically what this case case basically ah what this condition basically then means is that

so we have one at the center of the first circle and we have o2 as the center of the second circle and their point where they touch each other

so these two circles touch each other at exactly one point which is the point p and this point p lies on the straight line joining the centers this is what it means and it lies somewhere in between the straight line joining the centers

so in between one and o two

so we have something like this o one o two this is the point of contact p and further if we lets say at p if we draw a perpendicular

so at p we draw a perpendicular to this straight line one o two

so this perpendicular is this blue line then it is clear that the shortest distance between any point on this blue line and this centre one of the first circle will be the perpendicular distance from the centre one to this straight line and that perpendicular distance is obviously one p because we have constructed this line to be at ninety degrees to one o two and because one p is the radius of the first circle this this distance one p is equal to r one now clearly if we take any other point on this blue line then the distance of that point from one has to be strictly greater than r one because the closest point to one was this point p and we are now choosing another point on the straight line which is not p

so hence it is clear that the distance of that point of any other point apart from p on this blue straight line will be more than this radius r one and therefore that point will lie outside this first circle s one equal to zero similarly we will show that any point it is very easy similarly it is very easy to show that by using similar arguments that any point on the straight line apart from p will lie outside this second circle also and therefore all points of the straight line

so and therefore the only point on the straight line which touches both the

circle is this point p and hence this straight line is nothing but the transverse common tangent to both these circles

so this this condition d one what we have just seen now is that if the two
so and in this situation if you remember our previous lectures we say that whenever this happens that the point of contact of the point where the two circles touch if it lies between the two centers on the same straight line

so if the point of the point where the two circles touch each other which is this point p lies on this straight line one o two and between one and o two then we said when we say that these two circles touch each other externally

so hence what we show is what we have shown is that if the two circles touch externally then it must be true that the distance between the two circles is r one plus r two

so this we have just shown now ah rigorously however what about the other reverse argument let us say that we are given two circles and it is said that the distance between the centers is equal to the sum of the radius

so this is the reverse argument

so we are it is said that the distance between centers is equal to the sum of the radius then does it mean that the two circles will touch externally at exactly one point p that is indeed true because if we start with this equation and if we put it ah in this expression here

so we know that cos of phi minus alpha is equal to r one square minus r two square plus d one o two whole square by two r two d o one o two now in this equation if we put this d o one o two to be equal to r one plus r two and do a little math what we will see is that this value will come out to be minus one and then again this will basically mean that beta is equal to zero will mean that the two circles will touch each other at p it is not very difficult to also ah visualize this because as beta goes to zero what will happen is that this arm and this r with will both start coming towards the straight line joining the centers but this can only happen if we move this circle outside slowly

so when we move the circle outside then what will happen is that this angle beta will start reducing till we have moved it

so further that there is exactly one point where these two circles will touch in a similar manner if we when this can be left as an exercise for the students in a similar manner if we if we equate the right hand of this equation in the right hand side of this equation equals to plus one because we have seen the minus one case earlier if it equals to plus one then it is we can show that from here we can show that the distance between the two centers is equal to r one minus r two in our case we have taken r one greater than r two

so therefore this is in fact equal to the absolute difference of the radius but then if since this is equal to one the only solution is phi equal to alpha and phi equal to alpha will correspond to if we go back to this figure when phi is equal to alpha it corresponds to beta equal to pi

so beta equal to pi means that this this circle is moving inwards

so that this this is moving like this

so you can show that through examples

so one scenario is when the circles are touching ah at the the circles are intersecting at two points which is like this in which case this angle is beta now if this smaller circle moves further inside then will have a scenario something like this

so this will be the centers this will be the line joining the centers and then

so now this angle beta will increase

so this will be the angle beta now earlier it was acute now it has become obstures because this circle has moved inside

so and and when it

so happens that the second circle moves exact
so much inside that it it just touches the bigger circle at just one point in that case what will happen is that this
so this was the point p
so in that case what will happen is that this point p will come here and it will
so happen that one o two and p will be on the same straight line because this when beta becomes pi this this triangle one o two p will collapse into a straight line but then the difference from the minus one case is that in the minus one case the triangle o 1 o 2 p collapsed into a straight line which was one p o 2
so when we had the minus 1 case if we see the triangle one o two p collapsed into this straight line o one p o two with p in between one and o two because p was in between one and o two that is why we concluded that and because p was between one and o two and p is the point of contact of the point where the two circles touch
so that is how we concluded that the two circles must be touching each other externally but now for this plus one case what we see is that the triangle collapses into a straight line which is o one o two p
so clearly this can only happen if the circle the smaller circle touches the bigger from inside because the point of contact or point where the two circles touch is not lying in between the two centers it is on the same straight line but when extended forward when we take this straight line joining the center and if we produce it further then it actually meets p
so p is outside the line segment one o two although it lies on the same line but it is not a part of the line segment one o two and that helps us to conclude that the two circles must be touching each other internally and this is the condition for that and vice versa also it is very easy to show if we if we if we forget you know if we just look at this equation except you know we had put it two equal to one we had substituted this to be equal to one for the case where we wanted to see what happens because we were interested in the two scenarios where we have only one solution because when we have only one solution of phi it basically means that the two circles intersect each other at only one place because if we ah if we go back to this slide if we have different values of phi's each different value of phi will correspond to a different point p because if we change phi then the x and y coordinates will change which means that we get a different point of intersection but we saw that in some special scenarios where this where
so we saw that in special scenarios where this right hand side is either plus one or minus one in such cases there is only one solution of phi or one value of five we solve the equation there are not two values there is exactly one value and clearly one value of five will correspond to exactly one point where the two circles will meet which basically means that the two circles will touch each other at that point and then we can also for this case we can you know we start with this condition if we start with this condition then and if you put this value over here then what we will see that this right hand side will equate to plus one
so ah which basically means that if even if this condition is true then that also means that the two circles touch each other internally
so what we show i have shown now is that if d the the distance is equal to the absolute difference then the two circles touch each other internally and just prior to that we have shown that if the two circles touch internally if the two circles touch each other internally then it must be true that d o one o two is equal to the absolute difference and something similar we had shown for the case

where they touch externally and then we had also seen in our previous lecture that the two circles are going to intersect each each other either at one or two points
 so they will intersect at exactly two points if this right hand side has a magnitude which is less than one
 so and that we had analyzed in our previous lecture
 so so if the magnitude is less than one then the two circles intersect at exactly two points and if this value has a modulus equal to one then two circles touch each other and of course if this value the absolute value is greater than one then there is no solution of phi if this value is greater than one if the absolute value of this is greater than one then clearly there is no solution of five because the the the range of the cosine function is between plus one and minus one which basically means that the since there is no solution of phi it basically means that the two circles do not touch each other or neither do they intersect each other the two circles do not intersect and then a little bit of algebra for this case we ah showed that there were two cases corresponding to you know whether they touch externally or internally and then this ah particular case is is not very difficult to analyze and i think in the previous lecture we had said that if d o one o two is less than r one plus r two and if it is greater than the absolute difference if
 so so this ah this first case over here is exactly equivalent to this condition
 so we take up the a little example just to illustrate what we did what we have
 so far done in this lecture
 so let us say that we have two circles
 so let this be the coordinate axis x and y
 so so we have one circle whose center is at the origin and whose radius is let us say three units
 so the circle is something like this
 so this is the first circle and let us say we have another circle s two whose center is at this point which is five comma zero and whose radius is also let us say three units
 so so this is the other circle both of them have the same have the same radius but the centers are at different centers have different coordinates
 so they intersect at these two points p and q and if we were to write down
 so for this these two triangles this is o 1 o 2 and this is p
 so this is the triangle o one o two p that we get r one is three r two is three and d one o two is five
 so in this case for this case what we will get is that if we follow the same analysis that we are done for the general case then what we will get is that cos of phi minus alpha will be equal to
 so ah here ah our alpha is such that cos of alpha
 so this point is we had earlier represented the center of the first circle by a comma b center of the second circle by c comma d radius of the first circle by r one and radius of the second circle by r two and the distance between the two centers by d o one o two which is five in this case the radius are both three in this case a b is the origin c d is five comma zero ah cos of alpha was c minus a by d one o two which ah for this particular example is going to be c minus a is five
 so five upon five will be one
 so cos alpha is one n sine alpha is clearly zero because sin alpha was d minus b upon d one o two
 so that is what we have for this example and cos of phi minus alpha was ah we had the equation that it is equal to r one square minus r two square plus d over one o two square by two r two d one o two

so this will be equal to ah three square which is nine minus three square plus five square by two into three into five which will come out to be equal to minus 5 by 6.

so and then what we can do is that we can go back to our graph
so of course from here we also see that since cos alpha is one and sin alpha is 0 it follows that alpha is nothing but equal to 0 degrees and therefore cos phi minus alpha is nothing but cos phi itself and to find this solution or basically now that we want to find the coordinates of these two points what can simply be done is that we have to draw a green horizontal line with a displacement of minus 5 by 6 from with respect to the x axis but parallel to it
so that is going to be something like this
so this is five by six on the negative side
so we see that this minus five by six this horizontal ah line minus five by with at a displacement of minus five by six from the x axis and parallel to the x axis cuts or intersects this curve for cos phi minus alpha at two points and therefore these are the two solutions of
so these are the two values of phi which will give us cos of phi equal to minus cos of phi minus alpha to be equal to minus five by six
so therefore for this case ah what we have is that phi minus alpha will be equal to
so there will be two values
so because it is minus five by six
so and in this case alpha equal to zero
so we have to basically
so find the solutions to this equation also phi equal to minus five by six
so the the of course one value will be given by phi equal to cos inverse of minus five by six and this value of 5 will belong to the interval 0 to pi
so so this first phi angle is essentially this angle
so this is equal to cos inverse of minus five by six which belongs to which is between 0 and 180 degrees and the other value of phi will be equal to 2 pi minus this first value
so let this be phi one this will be phi two because we saw that there will be two solutions
so we will denote this one by phi one and this one by phi two
so phi one is equal to cos inverse of minus five by six
so and phi two will be equal to two pi minus cos inverse of minus five by six and this angle 2 pi minus cos inverse of 5 by 2 pi minus cos inverse of minus 5 by 6 is nothing but let us draw with
so this other phi 2 corresponds to this value here and that i will denote with this
so that will basically correspond to this angle
so from
so from from this line to all starting from here all the way till this line
so this angle in green is nothing but two pi minus cos inverse of minus five by six and as you can see this phi two will correspond to
so now if we
so ah now basically by just by using this different values of phi one and phi two and using the polar representation of the point of point where the 2 circles touch we know that x is equal to c plus r two cos phi
so let us say phi one and y is d plus r two sine phi one
so so
so when we put phi equal to phi one in the polar form we get these two points let us denote them by x one y one

so this will corresponds to

so phi equal to phi 1 will correspond to this point p because this angle is phi 1 basically

so so this is phi 1 and this other angle shown in green is phi 2.

so the coordinates of this point p which is this point is x 1 comma y 1 which will be given by ah this equation and this is not very difficult to calculate because we know all these values here we already know phi one we know r two equal to is equal to three we know that c is equal to five similarly we know d is equal to zero r two is ah three and since we know phi one we can compute sin phi one

so we can basically compute the coordinates of this point similarly to calculate the coordinates of this point q which let us denote ah by x two y two

so the coordinates of this point q x two comma y two can be calculated in a similar can be calculated in a similar manner but it is just that instead of phi 1 we will have phi 2 there where phi 2 is 2 pi minus cos inverse of minus five by six

so to be precise will have x two equal to c plus r two cos phi two and y two will be equal to d plus r two sine phi two phi two is this angle

so this was just two just a simple illustration of how we can use the techniques developed in this lecture to the technique that we developed could be first used was first used to essentially rigorously prove the conditions under which the two circles touch each other at exactly one point and as a byproduct we also see that the same technique can be used to find the coordinates of the point of intersection of these two circles and although in this example ah alpha was equal to zero but in in general alpha need not be equal to zero but even in that case its not very difficult because what we will instead get here is that

so instead of this phi here we would have had 5 minus alpha here we would have had phi 1 minus alpha and here we would have had phi 2 minus alpha

so if alpha was not 0 then the the solution would have been phi 1 equal to alpha plus cos inverse of minus 5 by 6 and phi 2 equal to alpha plus 2 pi minus cos inverse of minus five by six

so in the next lecture ah we will start a new topic called family of circles which will be something similar to what would have been discussed in the topic on family of straight lines thank you you