

welcome to lecture 10 on circles in one of the previous lectures we had discussed about the direct common tangents and transverse common tangents between any two given circles and we had specifically considered different cases in which one of the case was where the circles intersect each other and given the equations of the circles we had also told how to find out whether two circles are intersecting or not but then for those conditions we had not given any proof

so to be precise what we had said was that suppose if we have two circles s_1 one having equation $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and another circle s_2 having equation $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

so so we are given these two equations and then we are asked to find ah and then we ah said that these two circles will intersect each other if and only if the distance between their centers

so let their centers be

so the center of the first circle s_1 denoted by O_1 has coordinates $(-g_1, -f_1)$ and the center of the second circle O_2 is $(-g_2, -f_2)$ the radius of the first circle $r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$ and similarly the radius of the second circle s_2 is equal to $\sqrt{g_2^2 + f_2^2 - c_2}$

so just given this much information we said that if the distance between the centers

so if the distance between the centers which is square root of

so the distance between these two points which is square root of $(g_2 - g_1)^2 + (f_2 - f_1)^2$

so we said that if this distance is less than or equal to the sum of the radius or if it is greater than or equal to the difference of the the absolute difference of the radius of the two circles

so if this is true

so if this condition is true then we said that the two circles intersect each other we also said that if $d_{O_1O_2}$ which is the distance between the centers is equal to $r_1 + r_2$ then the two circles touch each other externally at exactly one point

so this is the case where we have something like this

so so this could be s_1 and this could be s_2 and they touch each other at exactly one point here

so let that point be p

so these are the centers O_1 and O_2 and then we had also said that the straight line joining the centers will also pass through this point p where the two circles touch each other and then we also said that if $d_{O_1O_2}$ is greater than $r_1 + r_2$ then the two circles do not intersect

so so this case is something like this where we have the first circle and the second circle and they do not intersect with each other

so this is this condition and then we had also said that if the distance between the centers is equal to the absolute difference between the two radius r_1 and r_2 then the two circles touch each other internally

so what we mean by this is

so this could be the first circle s_1 with center at O_1 and then we could have the other circle s_2

so this is s_2 with center O_2 and these two circles

so the circle s_2 touches the circle s_1 from inside that is why we said enter internally at just one point p and in this case the distance between the two centers is the absolute distance between the radius and then of course

we also had discussed the we had also discussed the last case where the distance between the centers is strictly less than the absolute difference between the radii in which case again the two circles of course they do not intersect they do not and further that one of the circles is going to be completely inside the other circle

so so we have a situation like this where let us say we have circle s_1 with center o_1 here and then we have this circle s_2 with center o_2

so these two circles do not intersect and further that one of the circles s_2 is completely inside the other circle s_1

so ah the main discussion of this lecture is going to be focused on rigorously deriving these conditions

so we will show that the two circles any two circles will intersect each other if and only if this condition is satisfied that is what that means is that if this condition is not satisfied then the two circles cannot intersect each other and then as special cases will also show that if the and this all of this will be proven rigorously because we had not done ah that in the lecture where we were discussing ah the derivation of the common tangents and then we will also show this special case where if the distance between the centers is equal to the sum of the radius then they will touch each other at exactly one point because when they intersect each other they will be actually intersecting at two different points but as a special case when the distance is equal to the sum then they touch each other externally by externally what what i mean is that the two circles are not inside each other

so for example s_2 is outside s_1 and it is touching s_1 from outside at this point p and then of course we will also show the rigorously derived this case also where the one of the circles touches the other internally

so let us start with seeing exactly under what circumstances will it

so happen that two circles will intersect with each other

so let us say that we have these two circles

so and

so we have circled s_1 and we have circle s_2 and let us say they intersect at these two points and let the coordinate of one of the points of intersection be x and y

so this is the center o_1 of the first circle whose coordinates i denote by a and b a comma b o_2 is is the center of the second circle whose coordinates are denoted by c and d c comma d and of course this the length of the straight line is the distance $d_{o_1 o_2}$ between the two centers

so since s_1 and s_2 intersect at this point x comma y this point x comma y lies on both the circles now to solve this problem we are going to use the parametric form of or the parametric equation of a circle

so if you remember the parametric equation of a circle then any point lets say x and y on the circle can be written like this

so the x coordinate can be written as x equals to the centre the x coordinate of the center of the circle plus the radius of the circle times \cos of sub angle θ

so this angle usually this angle is the if we draw a line through the center of this circle which is parallel to the x axis let us say

so this green dotted line then θ is simply the angle taken in the anticlockwise direction taken in the anticlockwise direction from the x axis x axis to the to this radius which joins one the center with this point x y

so this this this entire angle

so this angle is θ

so then using the parametric form we have the x coordinate of any point on the circle written as $a + r_1 \cos \theta$ and the y coordinate will be equal to

the y coordinate of the center which is b plus the radius times sine of the same angle theta now since this point x comma y also lies on the second circle s two we can write x and y in terms of the parametric form of the second circle s two also in that case x is equal to c plus r two cosine of some other angle phi because now the angle will be different because now we are trying to express this the coordinates of this point x and y with ah with in parametric form with regards to the second circle s2 now we if we again for to show phi what we have to do is we have to draw a line parallel to the x axis through the center o2 and then let us say we have this radius joining the center o2 with this point x y and then the angle from the horizontal or from the angle of this radius with respect to the x axis taken in the anticlockwise direction

so we have to see by what angle should this should this horizontal x axis be rotated in the anticlockwise direction

so that it coincides with this radius

so that angle is precisely this angle

so this much and i will denote it by phi

so then since this point x comma y lies on both the circles

so we initially express the coordinates x and y in terms of the parametric form in as in parametric form with regards to the first circle and since the same point also lies on the second circle

so we express the coordinates again in parametric form with respect to the second circle

so x is this and y will be d plus r two sin phi now if i equate this and this what i end up getting is that a plus r one cos theta is c plus r two cos phi

so at this point without loss of generality let us assume that

so without loss of generality let us assume that r one is greater than or equal to r two

so if r two is greater than r one then the same type of proof will follow is just that the rules of r one and r two will get reversed

so with this assumption by equating this and this what we actually get is r one cos theta is equal to c minus a plus r two cos phi and similarly if we equate this and this we get r one sin theta is d minus b plus r two sin phi

so if ah r two were to be greater than or equal to r one then we would have written this whole thing differently then in that case we we should have written r two cos phi is equal to a minus c plus r one cos theta and r two sine phi is equal to r one sin theta plus b minus d

so and then the remaining proof is exactly similar

so that is why we said without loss of generality now we square both these equations both the sides and add them up what we get on the left hand side is r one square cos square theta plus r one square sin square theta

so this is the left hand side on the right hand side we get c minus a plus r two cos phi the whole square plus d minus b plus r two sin phi the whole square because cos square theta plus sin square theta is equal to one the left hand side simplifies to r one square

so we have

so we have this equation and if we now expand the right hand side what we get is c minus a whole square plus d minus b whole square plus r two square cos square phi plus r two square sin square phi plus two r two c minus a cos phi plus two r two d minus b sin phi now this is nothing but r two square because cos square phi plus sin square phi is one

so we get r one square is equal to and this quantity is nothing but the distance between the two centers a b and c d

so this quantity is nothing but therefore this is equal to the square distance sorry

so this is the square distance between the two centers

so for therefore we have r_1^2 equals the square distance between the centers plus r_2^2 plus $2r_1r_2 \cos \phi$ plus d^2 minus $2d(r_1 \sin \phi + r_2 \cos \phi)$ and then we do a little manipulation here

so just this term we multiply and divide by d^2 and this becomes $\frac{r_1}{d} \cos \phi + \frac{r_2}{d} \sin \phi$ plus $\frac{d^2 - r_1^2 - r_2^2}{2d^2}$ now if we realize we know that $\frac{d^2 - r_1^2 - r_2^2}{2d^2}$ is $\frac{c^2 - a^2 - b^2}{2cd}$

so essentially we have something like this $\frac{c - a}{d} \cos \alpha + \frac{d - b}{d} \sin \alpha$

so we have a right angle triangle whose hypotenuse is of length d and the other two sides are $c - a$ and $d - b$ let us say and therefore it should be and we can in fact it is not very difficult to show that right angle triangle here

so if we go back to our figure

so this right angle if you draw a perpendicular from this centre O_2 to this green dotted line which is parallel to the x axis let us say this perpendicular then this is the right angle triangle that we are talking about because in this right angle triangle the hypotenuse is of length d this length is $c - a$ this is $d - b$ and let us denote this angle of the right angle triangle by α

so we have this angle to be α and therefore $\frac{c - a}{d}$ is $\cos \alpha$ and $\frac{d - b}{d}$ is $\sin \alpha$

so we will use this now in this equation

so therefore what we have is that $r_1^2 = d^2 + r_2^2 + 2r_1r_2 \cos \alpha \cos \phi + 2r_1r_2 \sin \alpha \sin \phi$

so this is using the $\cos(a - b) = \cos a \cos b + \sin a \sin b$ formula this is equal to $\cos(\phi - \alpha)$ and therefore from here we get that $\cos(\phi - \alpha) = \frac{r_1^2 - d^2 - r_2^2}{2r_1r_2}$

so if we

so let us since our main objective now is to find the value of this ϕ because if you remember the original problem was that we were given these two circles and we are supposed to find out the points of intersection now since we use the parametric form finding the points of intersection is the same as finding these angles θ and ϕ because once we find θ and ϕ we can obviously find x and y but then this θ and ϕ should be such that this simultaneously solve these two equations these are trigonometric equations

so we have two unknowns θ and ϕ which must be found out and all the others are all the other variables here are known to us because r_1 is known r_2 is known the coordinates of the 2 centers a b and c d are also known and that is what we have been trying to do in the last few slides and we have reached at this point where since everything else

so this right hand side is completely known to us α is also known to us because if we recollect \cos and \sin of α are trigonometric ratios of known quantities

so α is also known to us and therefore we should be able to find out ϕ and once we have found out ϕ we can plug that value of ϕ back into these two equations and we can easily find out θ and once we know θ and ϕ we can find the we know the coordinate of the point of intersection of these two circles

so to solve this let us draw the graph of $\cos(\phi - \alpha)$ versus ϕ

so at $\phi = \alpha$ $\cos(\phi - \alpha)$ will have the higher maximum value of one at $\alpha + \pi$ by 2 when $\phi = \alpha + \pi$ by 2 \cos of ϕ

minus alpha will be zero at phi equal to zero the value is cos of minus alpha which is let us say this value at alpha plus pi the value is going to be the value of cos of phi minus alpha is going to be minus one which is here at alpha plus three pi by two

so when phi is alpha plus three pi by two cos of five minus alpha is again zero so it is again here and then let us say that we only plot till two pi because it is enough since these function cos of phi minus alpha is a periodic function of phi periodicity being two pi it is enough to draw the graph between zero and two pi because for all other intervals

so for the interval from two pi to four pi the graph is going to be exactly similar similarly the graph from minus two pi to zero will also be the same as the graph for zero to 2 pi

so the graph obtained will look something like this then it goes to 0 here and then minus 1 and then back again to 0 here and at at two pi it is basically going to complete one full circle or one full ah one full two pi rotation and therefore this value and this value will be the same and then lets say that

so now it is clear that if this right hand side has a modulus greater than one then clearly there is no solution and that is precisely the case where what that means is that since there is no solution or basically there is no solution to this this trigonometric equation in that case where the absolute or or the value of the absolute value of this right hand side is more than one which means whether either it is greater than one or it is less than minus one

so in that case since there is no solution to this these trigonometric equations it is clear that the two circles wont intersect

so the other case is when the modulus of this absolute value of this right hand side is less than one in that case if the value is let us say if the absolute value is strictly less than one

so that is the case where

so let us say that the value is strictly less than one

so let us say the value is something like half

so then to find out the solution what we do is we

so let us say this is this this value is equal to this much

so this right hand side is equal to this much and therefore we draw a horizontal line parallel to the x axis

so sorry this this red will curve will only go till here because this and this should be same and therefore ah what we see is that for for any value which is strictly less than one it is very easy to see that there will be actually two different solutions or two different values of phi which will satisfy this equation

so we could takes any other value also

so we could take some other value let us say let us say this value which is let us say minus one by four

so if this if this right hand side is minus one by four then to find the solution will draw a horizontal line parallel to the x axis whose vertical distance from the from the x axis is one by four but it is on the negative side on the is on the negative side

so so basically this green line

so when this is minus one by four the two solutions are given by the point of intersection of this green line with the curve of cos phi minus alpha and those two points are this

so these two one

so one can easily see that ah for any value of this right hand side which has an absolute value less than one there will exist two distinct values of phi or two distinct solutions for phi and for each such solution of phi if we put it

back in this equation here then we will get a unique value of theta corresponding to that phi

so that will be one theta and phi pair now because there are there are two different phi's for such cases where the right hand side has absolute value less than one it is clear that there will be two different theta phi pairs what that means is that there will be two different points of intersection for example as shown in this figure here now for this case where where the absolute value

so this is the case where which we are dealing with right now right now the absolute value of the right hand side of the equation on the previous slide of this equation

so the if this right hand side has absolute value strictly less than one so this is the case that we are considering and we have argued that in this case there will be two points exactly two points where the two circles will intersect but then ultimately what we want to show is that we want to show that this condition implies and is also implied by the condition that the distance is less than the sum of the radius and greater than the absolute difference

so this is what we should get starting from here both ways

so if we start from here we should get this and this should also imply this to but if you want to see that we have to go back to our the the equation before this equation which is written here

so if we again see this equation r_1^2 or rather if we if we see this equation the same equation again it reminds us of the cosine formula the cosine law this reminds us of the cosine law because if we remember the cosine law what we had was that let lets say we have a triangle with sides r_2 , r_1 and d and lets say that the angle between the sides of length r_2 and d is beta then we know that cosine of beta is nothing but $r_2^2 + d^2 - r_1^2$ upon $2 r_2 d$ and this right hand side is very similar to this side except that we have to negate this except for a negative sign

so if we take minus of this right hand side will exactly get this

so what that means is that if we again go to our the starting slide then let us try to see where exactly is this angle i mean beta that we are talking about

so we if we go back here what we see is that let us look at

so let us this let this point be p and let us look at the triangle $p_1 o_1 p_2$ then we see that this side $p_1 o_1$ is of length r_1 , $p_2 o_1$ is of length r_2 and $p_1 o_2$ is obviously of length d

so this is the triangle that we had drawn on the just a few minutes back and then we were talking of the angle between this r_2 length side side of length r_2 and this side of length d

so this is what this angle is what

so this angle is what we were calling as beta but then its not very difficult to find beta because if we see this is nothing but 90 degrees $\pi/2$ and this angle was alpha

so if you look at this right angle triangle here this is alpha and therefore this angle here is $\pi/2 - \alpha$ now if we sum all these one two three and four all these four angles we should get 2π

so 2π is equal to

so let us start in the anticlockwise direction starting with phi

so phi plus and then beta plus $\pi/2 - \alpha$ plus $\pi/2$

so from from this equation what we will get is that this angle beta is equal to $\pi - \phi + \alpha$ now if we if we if we see this let me draw this triangle over here

so so this is beta this r_1 , r_2 and this is d

so if we apply the cosine law on this triangle what we will get is cos exactly

what we had written here but then we have shown just on the previous slide that β is equal to $\pi - \phi + \alpha$ and therefore $\cos \beta$ is nothing but $\cos(\pi - \phi + \alpha)$ which is nothing but

so $\cos \beta$ is equal to $\cos(\pi - \phi + \alpha)$ which equals $-\cos(\phi - \alpha)$ and we get a negative sign here

so i mean

so this is equal to this which basically means

so from this we will get this equation

so what we have here is basically that

so this equation here that we got is nothing but the cosine law applied to this triangle one o two p now from the triangular inequality

so what we have is that if this condition is true if this condition is true then in that case the two circles will intersect at two points

so this is the point of intersection p and we have this triangle here one o two p and for this triangle triangular inequality must get satisfied and since the triangular inequality must get satisfied it must be true that $d_{12} < r_1 + r_2$ otherwise we cannot have a triangle on the other hand if we know that $d_{12} > |r_1 - r_2|$ if we have if we

so this is one thing the other thing is that

so this is one condition that we get because of the triangular inequality the

so but this is only one of the time because totally will be having three inequalities

so the other ah triangular inequality is when we put r_1 on the left side and we say $r_1 < d_{12} + r_2$ or $d_{12} > r_1 - r_2$ now in this case since without loss of generality $r_1 > r_2$

so this is nothing but the same as its absolute difference because $r_1 > r_2$

so this is true because $r_1 > r_2$

so therefore this ah second triangular inequality implies this and the third triangular inequality will not give us anything meaningful because the third will be $r_2 < d_{12} + r_1$ which is anyways true because $r_1 > r_2$ and the distance between the two centers is positive

so this will not give us anything meaningful

so what we saw was that if the right hand side of this triangular of of this equation here is if the if the absolute value is less than one then we argued that there will be two points of intersection and then we also showed that using we showed that this equation corresponds to the cosine law for one of the triangles in the first figure and then applying the triangular inequality to that triangle we have showed that these two conditions must be satisfied if the two circles are intersecting at two points then this must be true

so we have shown only one way implication that if the two circles intersect then these two should be satisfied but the reverse is also true because if these two conditions are satisfied this third condition is anyways true because $r_1 > r_2$ and the distance between the centers is positive and therefore since

so therefore we have three numbers three positive numbers r_1 , r_2 and d_{12} which satisfy all the three triangular inequalities and therefore they must one can always construct a triangle with r_1 , r_2 and d_{12} as its side and then we can just take the argument backwards and if we take the argument backwards it is not very difficult to show that they are in the two triangles will intersect and they will exactly intersect at this point p because since these three numbers r_1 , r_2 and d_{12} satisfy this triangular inequality a triangle must exist with two of his vertices as one and o two because d_{12} is nothing but a distance between the two centers right and

with another vertex whose distance from o_2 is r_2

so this vertex must lie on the second circle because the second radius of the second circle is r_2 and the distance of this point p is r_2

so this point we must lie on this circle and similarly since this other the third length is r_1 the same vertex of this triangle p must also lie on the first circle and therefore since it lies on both the circles this has to be a point of intersection

so let me again repeat the the ah the other argument

so so

so what we firstly showed was that using triangular inequality we showed that ah this condition implies that these two are must hold now we will show the the the backward argument that if we if we are only given these two conditions then also these two conditions will imply that the two circles will intersect that is because we are given these two conditions this condition is anyways true because r_1 is greater than r_2 r_1 is greater than or equal to r_2 and the distance between the center is positive now given these three conditions we realize that this c condition nothing but the triangle they seem to be the three equations of triangular inequality

so we basically have three numbers with positive numbers r_1 r_2 and d $o_1 o_2$ which satisfy the triangular inequality and therefore we must be able to construct a triangle whose the sides have length r_1 r_2 and d $o_1 o_2$ now d $o_1 o_2$ is nothing but the distance between the two centers

so we choose two of the vertices of this triangle that we are constructing to be o_1 and o_2 and the third vertex we can choose it to be a point which is at a distance of r_2 from the second the center of the second circle and at a distance of r_1 from the center of the first circuit but then we know that the radius of the second circle is r_2 and therefore this point must be on the second circle similarly we know that the radius of the first circle is r_1 and therefore this point must also be on the source circle and since it is on both the circles it must be the one of the points of intersection which means that the two circles are intersecting

so what we have

so far shown in this lecture is that if the two circles intersect at two points

so what i mean to say is that if there are two distinct solutions or two distinct solutions of ϕ to this equation which will happen only when this right hand side has absolute value less than one

so if we assume that condition that if the right hand side here has absolute value less than one in which there are two distinct solutions of ϕ which basically means that we are assuming that the two circles intersect at two different the intersect at two different points

so if the two circles if we assume that the two circles intersect at two different points then starting from here and using triangular inequality what we have shown is that the distance between the two centers must be strictly less than the sum of the radius and also that the distance between the centers must be strictly greater than the absolute difference between the two radius and then after that we also showed that we also showed the reverse argument we showed that if we start with the if we start with the assumption that the distance between the two circles is less than r_1 plus r_2 and it is greater than the absolute difference between the two radius if we start with that assumption then also we showed that we argued that it must be and maybe in the first 15-20 minutes of the next lecture we should be able to finish the remaining argument

so that what the remaining cases are basically the cases where we have the absolute value to be equal to one or the absolute value to be greater than one for example

so we will take those cases in the next lecture

so we will finish that in the first 15 20 minutes of the next lecture and the remaining part of the next lecture will start a new topic on family of circles
thank you you

Prutor@IITK