

welcome to lecture nine on circles in the previous lecture we had discussed about we had solved some problems related to common tangents of two given circles

so in this lecture we will talk about the angle of intersection between any two given circles and then we will move to finding the condition under which any two given circles are orthogonal to each other will also define something that is known as the radical axis between any two given circles

so let us start with defining the angle of intersection of two given circles

so suppose the equation of two circles is given to us and let us say that the two circles are intersecting each other

so obviously the angle of intersection is only defined for two circles which intersect each other if the two circles are not intersecting with each other then the angle of intersection is not defined in that case

so let us say that these are the two circles which intersect with each other

so this is the first circle S_1 this is the second circle S_2

so let us say that the centres of these two circles are at O_1 and O_2 the equation of the first circle is let us say $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

so this is S_1 and the equation of the second circle S_2 is $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

so these two circles intersect at these two points now let us draw a tangent to the first circle

so a tangent at this point of intersection

so the tangent will look something like this

so essentially this is going to be 90° degrees similarly let us also draw a tangent to the second circle at the same point of intersection

so that tangent is drawn in red here

so that might look something like this

so this is the straight line tangent to the second circle at this common point of intersection I will call it T_2 and the tangent to the first circle at the same point of intersection I will call it by T_1 and then let this angle between these two tangents be θ

so this angle between the two tangents to the two circles at this point of intersection

so this angle is what is known as the angle of intersection between the two circles

so now given the equation of the two circles we should be able to find this angle of intersection θ

so for that let us denote this point of intersection by A and let us also connect O_1O_2 by a straight line

so now what we have is a triangle O_1AO_2

so this is a triangle that we have

so this the length of the side O_1A is equal to r_1 which is the radius of the first circle S_1 well r_1 of course is equal to the square root of $g_1^2 + f_1^2 - c_1$ where we already know the values of g_1 , f_1 and c_1 because the equation of the first circle is given to us similarly we can find this length O_2A which is actually the radius of the second circle and that can again be found from the since we already know the equation of the second circle we can know the values of g_2 , f_2 and c_2

so r_2 will simply be square root of $g_2^2 + f_2^2 - c_2$ and then of course since we already know the coordinates of the center

so the coordinate of the first center the center of the first circle is $(-g_1, -f_1)$ and then the coordinates of this point O_2 which is the center of the second circle is $(-g_2, -f_2)$ and then the

distance between them which is

so one o two the distance between the two centers is given by square root of g^2 minus g^2 whole square plus f^2 minus f^2 whole square

so now what we have here is we have a triangle one a o2 and we exactly know the lengths of its three sides and therefore now it should be possible to also find the three angles of this triangle but then we are actually asked to find this angle theta what we also realize is that since t2 is a tangent to the second circle this angle is also 90 degrees

so now if we look at this point o we look at this point o

so we first have this angle which is 90 then we have theta and then we have this angle which is 90 right and then finally we have this angle o1 a o2 since the sum of all these angles should be 360 what we have is that

so the the first angle is 90 degree because of this

so that is the angle between this o 1 a and this tangent t 1 that is 90 degrees

so so pi by 2 plus and then we have the angle of intersection of the two circles which is this angle theta plus then again we have 90 degrees between this the normal o2 a

so the angle between o two a and t two is ninety degrees

so we have again pi by two and then plus angle o two a o one

so angle o two a o one

so sum of all this should be equal to three hundred and sixty degrees which is two pi and therefore from there we can say that angle o two a o one must be equal to pi minus theta we will write here pi minus theta now we will simply apply the cosine law to this angle angle o2 a o1 of this triangle

so by the cosine law cosine of this angle o2 a o one is equal to the sum of the squares of the two sides which are two sides which are adjacent to this angle or

so so the 2 sides are basically in this case r^2 square and r^1 square

so r^1 square plus r^2 square minus the square of the side which is opposite to this angle which is

so it will be minus off this this is the the distance between the two centers the square of that divided by two times the product of the length of the sides adjacent to this angle

so divided by two times $r^1 r^2$

so of course now from here ah taking it further now we already have expressions for $r^1 r^2$ and o one o two in terms of the ah because we already know the value of the $g^1 g^2$ and $f^1 f^2$ and $c^1 c^2$ since the equation of the two circles is given to us

so we should be able to exactly find out the cosine of this angle but we already know that since angle o two a o one is pi minus theta cosine of angle o two a o one is cosine of pi minus theta which is equal to minus cos theta but we also know that this is equal to from the previous slide we will just replace the expressions for $r^1 r^2$ and o 1 o 2

so these 3 expressions will be substituted here and there by we are going to get this cosine minus of cosine theta which is cosine of angle o2 a o 1 to be equal to

so r^1 square will be g^1 square plus f^1 square minus c^1 plus r^2 square will be g^2 square plus f^2 square minus c^2 minus of one o two whole square

so one o two whole square will be g^1 square plus g^2 square minus two $g^1 g^2$ then plus f^1 square plus f^2 square minus two $f^1 f^2$

so so this is the numerator and the denominator we have two times $r^1 r^2$

so it will be 2 times square root of g^1 square plus f^1 square minus c^1 times square root of g^2 square plus f^2 square minus c^2 and this

so essentially what we have is that \cos
so this is minus of $\cos \theta$ therefore $\cos \theta$ where θ is the angle of
intersection between the two circles

so $\cos \theta$ will be equal to $\frac{c_1 + c_2 - 2g_1g_2 - 2f_1f_2}{2\sqrt{g_1^2 + f_1^2} \sqrt{g_2^2 + f_2^2} - c_1c_2}$ and we
already know that since θ is between 0 and π

so it is clear that θ also is going to be within in
so θ is going to lie in the range 0 to π and
therefore the value of θ will be nothing but

so θ will be nothing but \cos^{-1} of this right hand side
so θ will be equal to \cos^{-1} of
so the argument of \cos^{-1} will be this expression now let us see the
condition under which two circles will be orthogonal

so when we say two circles are orthogonal
so two circles are said to be orthogonal if and only if the angle of
intersection between them is $\frac{\pi}{2}$ or 90°

so now let us see what is the condition under which or which the condition that
must be satisfied in order that two given circles are orthogonal to each other
that should not be very difficult from here because two if two circles are
orthogonal then this θ must be equal to $\frac{\pi}{2}$ but since \cos of $\frac{\pi}{2}$
is zero it is clear that for two circles to be orthogonal this right hand side
must be zero

so so therefore the condition is that two circles s_1 and s_2 whose
equations are given here
so two circles s_1 and s_2 are orthogonal if and only if this expression is
zero which basically means that $2g_1g_2 + 2f_1f_2 = c_1c_2$

so let us take a couple of questions to make this concept a little clearer
so here is the first question it is said that a circle s passes through the
point $(0, 1)$ and is orthogonal to these two circles
so let us say that the circle s has this equation and since it passes through
the point $(0, 1)$ this equation must be satisfied with $x = 0$ and $y = 1$
therefore we get $1 + 2f + c = 0$ also this circle
 s is orthogonal to both the circles

so since it is orthogonal to this circle we can use the condition for
orthogonality

so we can use this condition
so this equation can be written as $x^2 + y^2 - 2x - 15 = 0$ and the second circle is
 $x^2 + y^2 - 2x - 1 = 0$

so ah
so here let us say
so if we if we if we
so consider the equation here to be $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ then $g_1 = -1$, $f_1 = 0$, $c_1 = -15$
similarly for this second circle we if we consider this to be $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
then clearly $g_2 = -1$ and $f_2 = 0$ and $c_2 = -1$ since this
circle is orthogonal to the first circle we have since this circle s is
orthogonal to this first circle it must satisfy the equation of orthogonality
which is $2g_1g_2 + 2f_1f_2 = c_1c_2$ which is $0 = -15$

so so this is essentially $2g + c = 15$.

so this was the first equation this is the second equation ah similarly because the circle is also also also orthogonal to the second circle we have a similar type of equation $2g^2 + 2f^2 = 0$ plus $2g^2 + 2f^2 = 0$ which is also zero equals $c + c^2$ this is the third equation from this third equation we clearly get $c = 1$ and if we use that information in this first equation we will get $f = -1$ now since $c = 1$ if we use the same $c = 1$ information of $c = 1$ in this is the second equation we get $g = 7$

so then we have gotten all the parameters of this circle S and clearly the center of this circle S is center is $(-g, -f)$ which is $(-7, 1)$ because $g = 7$ and $f = -1$

so the center is $(-7, 1)$ which means that option c is correct and option d is wrong and the radius is equal to $\sqrt{g^2 + f^2 - c}$ will come out to be 7

so option b is correct one option a is wrong

so let us consider another problem

so here it is given that the circle passes through the point (a, b) and this circle let's say denoted by S and it cuts another circle $x^2 + y^2 = k^2$ orthogonally then the locus of the centre of the circle S is one of these four options that we have to find

so let the equation of the let us say that the center of circle S be (p, q) where the coordinates of the centre of circle S be (p, q) then the equation of this circle S will be $x^2 + y^2 - 2px - 2qy + c = 0$ now it is said that the circle S passes through the point (a, b)

so what that means is that this equation should be satisfied with $x = a$ and $y = b$ therefore $a^2 + b^2 - 2ap - 2bq + c = 0$ this is the first equation that we get and then it is also said that this particular circle S is orthogonal to another circle let us say S' whose equation is $x^2 + y^2 - k^2 = 0$

so we use the condition of orthogonality then what we get is

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

so so the g of the g_1 for this first circle is $-p$

so $2g_1g_2 + 2f_1f_2 = c_1 + c_2$ but the g_2 here is zero and then plus two times f_2

so $f_2 = -k$ times f_1 is zero equals $c_1 + c_2$

so $c_1 = c$ and $c_2 = -k^2$

so since these two circles are orthogonal this equation must also be satisfied

so this equation basically comes from what we had shown in one of the previous slides to be the condition for orthogonality between two circles

so from this equation it is clear that c must be equal to k^2 and therefore the equation of the circle S

so therefore if we use this fact in equation one what we get is that the coordinates p and q of the center of the circle S must satisfy the equation $a^2 + b^2 - 2ap - 2bq + k^2 = 0$ or in other words

so so what this basically means is that the coordinates p and q of the center of the circle S must always satisfy this equation and therefore the locus of the center of circle S is $a^2 + b^2 - 2ax - 2by + k^2 = 0$

so let us say that we

so this will be $a^2 + b^2 - 2ax - 2by + k^2 = 0$ the center

so let that be x minus two b times the y coordinate of the center plus k square is zero

so the locus of the center is essentially this equation which is actually a straight line equation because it is linear in both x and y and this is nothing but the first option which is option a

so let us move on to a new topic which is defining what is known as the radical axis of two given circles

so suppose that we are given two circles

so let us say this is circle s_1 and then we have another circle over here let us say s_2 now consider all those points such that we will only consider those points p such that the length of the tangent from this p to both the circles are equal

so so let us say that these are the two circles with centers O_1 and O_2 and p is a point such that the length of the tangent PA from this point p to this first circle s_1 is equal to length of the tangent from p to the second circle s_2

so will only consider those points for which PA and PB are equal in this case ah at least in appearance it does not look like PA and PB are equal

so if they are not equal then p is will not consider p as one of the points in which we are interested

so the locus

so there is not just one unique point which has ah the same distance whose length of the tangent to both the circles are equal there are infinitely many points and the locus of all these points as we will see soon is a straight line which is actually called the radical axis of these two given circles

so let us see how to derive the equation of this radical axis if we are given the equations of the two given circles

so suppose that we have two given circles s_1 and s_2 having centers O_1 and O_2 then we are only we are always considering those ah points only considering those points which have for whom the length of the tangent to both these circles are equal

so for example if a point p has to lie on the radical axis then the length of the tangent from p to the first circle let us say PA must be exactly equal to the length of the tangent from p to the second circle PB

so PB and PA must be equal

so for this to happen

so let us say that we are given the equation of the two circles to be as follows

so we are given the equation of two circles and we have to find the equation of the radical axis now suppose that there is a point p having coordinates x, y the center of the two circles are of course (g_1, f_1) and (g_2, f_2) now

so this has to be ninety degrees

so the length PA or the square length PA^2 is equal to one PA^2 square using the pythagoras theorem because this triangle O_1AP is a right angle triangle we have PA^2 is nothing but one PA^2 square minus one O_1A^2 square and this will if we do further calculations as we had already seen in one of the previous lectures the squared distance of this ah line segment PA will be equal to $x^2 + y^2 + 2g_1x + 2f_1y + c_1$ where x and y are actually the coordinates of this point p

so we have taken x and y to be the coordinate of this point p in that case this squared length PA^2 and PB^2 if you also recollect is called the power of this point p with respect to the first circle similarly the power of this point p with respect to the second circle will be PB^2 which is also

which is equal to $x^2 + y^2 + 2g_1x + 2f_1y + c_1$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2$ now for uh now since $PA = PB$ are equal it follows that the coordinates x and y must be such that this expression and this expression are equal that is so this two should be equal and from here we get that the coordinates x and y of this point P must satisfy the equation $2g_1 - 2g_2 = 2x + 2f_1 - 2f_2 = 2y + c_1 - c_2$ equals zero

so we see that all such points whose power with respect to two both the circles are equal the coordinates of all such points must satisfy this equation which is nothing but the equation of a straight line and this straight line is called the radical axis of these two circles

so we will get many points like that

so just like P will get another point probably here such that this length and this length are equal similarly there could be another point here such that the this length of this part of the tangent to the first circle and then this length which is part of the tangent to the second circle

so this and this length will also be equal

so there will be infinitely many such points and if you all join all these points we will get this straight line this straight line whose equation is this and this straight line is called as the radical axis of these two circles it is also not very difficult to see that if two circles were to intersect each other then we have already seen that the equation of the common chord

so we saw that in the last lecture or maybe the lecture before that that the equation of the common chord the common chord is the point join the straight line joining the two points of intersection and we saw that the equation of the common chord is nothing but

so this is the case where the two circles are intersecting with each other

so in one of the previous lectures we have already seen the equation of the common chord to be this particular equation but then this is nothing but the equation of the radical axis hence when two circle intersects the radical axis is nothing but the common chord we just have to extend it further in both the directions another case could be when two circles touch each other at one point and in that case one can show that the radical axis is nothing but the transverse common tangent between these two circles whose equation will be in that case will be the same as this equation also if we go back to this equation of the radical axis then it is not very

so the equation of the radical axis was this equation

so the slope is the slope of this straight line for the slope of the radical axis which is a straight line is $\frac{g_1 - g_2}{f_1 - f_2}$ and the slope of the line joining the centers one and two of the two circles is $\frac{f_2 - f_1}{g_2 - g_1}$ now if we take the product of these two slopes we see that the product is minus one what that basically tells us is that the radical axis is always perpendicular to the line joining the centers of the two circles now given that we know what is meant by the radical axis between any two given circles will now define what is known as the radical centre of any three given circles

so suppose we are given three circles like this

so let us say that the centers are at O_1 , O_2 and O_3 and let us say that the equation of the first circle is $S_1 = 0$

so so it's basically the equation of the first circle will be $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ similarly we will have similar equations for the other two circles and for the third circle

so this is S_3

so $x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$

so let us say that these three equations are given to us and we are asked to will basically define what is meant by the radical center of three circles

so we already know that the equation of the radical axis between the first and the second circle is a straight line

so this straight line need not pass through the center of the third circle this is this is just this just an example

so it need not pass through the center of the third circle

so the this this is the radical axis between the first and the second circle whose equation is basically $s_1 = s_2$ or $s_1 - s_2 = 0$

so from here $s_1 - s_2 = 0$ will be

so so that equation

so this equation will be simply what we have seen in the previous slide

so the radical axis between s_1 and s_2 will have the equation $2g_1 - d_1^2 x + 2f_1 - f_1^2 y + c_1 - c_2 = 0$ and similarly will also have a radical axis between s_1 and s_3 between the first and the third circle

so let that be shown by this green line here

so this is the radical axis between the first and the third circle let us say and the equation of this radical axis will basically be $s_1 - s_3 = 0$ which will be $2g_1 - d_1^2 x + 2f_1 - f_1^2 y + c_1 - c_3 = 0$ we will call it c in the following what we will show is that the radical axis between the second and the third circle will actually pass through this point of intersection of the radical axis of the two radical axis that we have already seen would be something like this

so what i have drawn in red is the radical axis between the second and the third circle

so in fact the three radical axis between the three pairs of circles are or all three of these are concurrent at one point which we have denoted by c and this c is then called the radical center of these three circles but we need to first show that the radical axis between the second and the third circle will indeed pass through the point of intersection of the first two radical axis in blue and green the equation of the radical axis between s_1 and s_2 which we had drawn in the blue line is similarly the equation of the radical axis between s_1 and s_3 is and the radical axis the equation of the radical axis between the second and the third circle is

so these are the equations of the three radical axis for the three different pairs of circles now we saw that these two radical axes were intersecting at a point c now from our lectures from the lectures on straight lines we already know that the equation

so suppose if we have two straight lines $l_1 = 0$ and $l_2 = 0$

so these are two straight line equations which intersect at a point c then we know that the family of straight lines passing through this point of intersection c is given by the general equation $l_1 + \lambda l_2 = 0$ where λ could be any real number we choose different values of λ will get different straight lines but then all these straight lines no matter what value of real value of λ we choose all these straight lines of this form will pass through the point of intersection of these two straight lines

so this we already know

so applying this to our case

so let us say that this is the first line $l_1 = 0$ and this is the

this is the second line l_2 equal to 0 and both of them intersect at this point c then clearly the equation of the family of straight lines

so any straight line which passes through this point of intersection c can always be written in this form $l_1 + \lambda l_2 = 0$

so l_1 is this plus λ times l_2 is this

so $l_1 + \lambda l_2 = 0$

so this for any λ any real λ this is also the equation of some straight line but this straight line which we have written here will pass through the point of intersection of the two radical axis which is this and this which we have denoted by c if we take λ equal to minus 1 then in then we get a specific straight line which is given by whose equation is given by

so clearly even if we take λ equal to minus one we get a straight line which will pass through c and that straight line is $2g_1 - d_1^2 - x^2 + 2g_2 - d_2^2 - y^2 + c_3 - c_2 = 0$ and if we simplify this straight line equation what we get is $2g_3 - d_3^2 - x^2 + 2g_2 - d_2^2 - y^2 + c_3 - c_2 = 0$ but then this equation

so this equation is the equation of a straight line which passes through that point of intersection c of these two radical axis

so the point of intersection c of these two radical axis but then this straight line is nothing but it is same as the third radical axis between the second and the third circle

so this equation is the same as this equation

so this is nothing but the radical axis between circles s_2 and s_3 and we already know that the point of intersection of the first two radical axis lies on this straight line and therefore it is clear that the radical axis between the third pair which is s_2 and s_3 will also pass through the point of intersection of the first two radical axis essentially what that means is that all these three radical axis are concurrent at one point c at a point c and this point is then called the radical center of the three circles in the next lecture we will discuss as to how to derive the equation of a family of circles for example the equation of all those circles which pass through the intersection of two given circles or the family or or the equation of all circles which pass through the intersection of a given circle and a given straight line

so we will see this in the next lecture thank you you