

welcome to lecture eight on circles

so in the last lecture we had derived the expressions for the equation of the common tangents to any two given circles

so in this particular lecture we will solve some problems ah on that topic and then we will quickly move on to a new topic which is ah basically defining something which is known as the angle of intersection between any two given circles will also derive the equation of the common chord of two intersecting circles

so let us quickly start with some problems on the topic that we discussed in the last lecture

so let us see this question first where it is asked to find the coordinates of the point at which the circles

so we are given two circles this is the equation of the first circle

so $x^2 + y^2 - 4x - 2y + 4 = 0$

so this is the first circle and the other circle is given by has equation $x^2 + y^2 - 12x - 8y + 36 = 0$

so we are asked to find the coordinates of the point at which these two circles touch each other also find the equation of the transverse common tangent of these two circles

so the problem statement is very clear we have two circles and it is said that they touch each other and in that case we are asked to find the coordinates of the point where they are touching each other and then further we are asked to find the equation of the transverse common tangent between these two circles

so ah to illustrate this let us try to do this geometrically

so let this be the y axis and let this be the x axis this is the origin

so the the first circle which is this circle this clearly has ah center which is at two comma one because ah this particular term is plus two g x and this is plus two f y

so g is minus two f is minus one and the center we know is at minus g minus f which is two comma one

so we'll denote this circle by c one and the radius is of course equal to $\sqrt{g^2 + f^2 - c}$

so that will be

so for this we know that g is minus 2 f is minus one c is four

so this will be $\sqrt{2^2 + 1^2 - 4}$ which is same as $\sqrt{2^2 + 1^2 - 4}$ minus four this will come out to be one

so given the center and radius we can plot the first circle

so the center is over here and the radius is one and

so since the radius is one the circle will appear something like this similarly we can show that for the second circle the center will be at six comma four and the radius will be four

so this is the center of the second circle and as the radius is four units the circle will be something like this similarly on this other side which i am not drawing then at least geometrically it appears that they are intersecting their it appears that they are touching each other at this point but it is also not very difficult to verify that

so if the circles if two circles are touching each other then it must be true that the distance between the centers has to be equal to the sum of their radius

so the distance between these two centers is equal to $\sqrt{6^2 + 4^2 - 1^2}$ which comes out to be five and if you remember the radius of the first circle was one and for the second circle was four

so we when they add them when we add them up we some of the radius is five

which is exactly equal to the distance between the two centers of the circles which means that these two circles are touching at exactly one point

so let this this is the line joining the two centers

so we were asked to find out the coordinate of this point where these two circles touch each other and that is not very difficult because suppose let us say that the coordinates of this particular point where the intersect is x y then it is clear that this x and y must satisfy both these equations because this point lies on both the circles and since it has to satisfy both the equations it must satisfy the difference of the equations also

so what i mean to say is that because this coordinates of the point of contact of these two circles x and y must satisfy the two equations the equations of the two circles which are as follows and therefore if we subtract it then what we get is

so this point x comma y must also satisfy this equation or we can simplify to $4x + 3y = 16$ this in fact this equation is

so it must be satisfied by this equation must be satisfied by this point now we know that the equation of this particular line joining the centers is known and we know that this point x y also lies on this line line joining the centers and therefore it must be true that

so the slope of this line is equal to $\frac{4 - 1}{6 - 2}$ which is $\frac{3}{4}$ which must also be equal to $\frac{y - 1}{x - 2}$

so the slope of this line must be the same as the slope of this line segment because essentially they are the they are segments of the same line $y - 1$ divided by $x - 2$

so therefore the coordinates x and y must also satisfy another equation and ah this equation we can write it down is $3x - 6 = 4(y - 4)$ which is

so essentially we have $3x = 4y + 2$ and then we just need to solve these two equations

so by solving these two equations will get the coordinates of this particular point here and that is not very difficult

so what we can do is you can multiply this equation by four we get we multiply this equation by three we get $9x = 12y + 6$ and then we add these two equations we get $25x = 70$ which means $x = 14$ upon 25 or which is the same as 14 by 5 and then the y coordinate is very easy because that we can just use this x value here and we can get the y coordinate

so the y coordinate will be $3y = 16 - 4x$ from here

so that will be $16 - 4 \times 14$ by 5 which is 56 by 5

so that comes out to be

so $3y$ will be equal to 24 by 5 .

therefore y will be eight by five

so the coordinates of the point of contact now the point where c_1 and c_2 touch each other is 14 by 5 comma 8 by 5

so the coordinates of the point where they touch each other is 14 by 5 comma 8 by 5

so but that only solves one part of the question because it was also asked to we were also asked to find out the equation of the transverse common tangent

so let me draw the

so the transverse common tangent is ah basically a tangent like this which is a tangent to both the circles

so the same tangent is a tangent for this first circle as well as for the second circle

so obviously that has to be perpendicular to this line joining the two centers

of the circle

so it will be something like this

so it will be perpendicular to this straight line drawn in black and it will also pass through this point of contact those coordinates we have just found out now

so this green line is basically the transverse common tangent and for finding the equation we do not have to do anything much because if we if we if we go back then this was the line equation of the line joining the centers of the circle and if you see the slope of this line slope is equal to three by four which we had already calculated here now the slope of the line perpendicular to this line joining the centers will be minus four by three because we know that the the the product of the slope of two perpendicular lines is minus one

so therefore we know that the slope of this must be minus four by three into and interestingly if we see this equation here this is also the same equation of some straight line because its a degree one equation in x and y

so the the this represents a straight line the slope of the straight line is also minus four by three and we already know that this point of contact satisfies this equation the point of contact is also on the tangent and therefore it is clear that the equation of this ah this green tangent or the transverse common tangent is nothing but this straight line equation because the the slope of this straight line equation is also minus 4 by 3 plus this point x y which is on the tangent is satisfying the coordinates of this point is satisfying this equation

so therefore the equation of the transverse common tangent to both the circles is $4x + 3y = 16$

so that finishes the solution to the first problem and here you might be wondering whether because what we saw in this problem was that interestingly the equation of the transverse common tangent in this in this scenario where the two circles touch each other came out to be nothing but the difference of the equations of the two circles but is that true in general or maybe it was just luck by chance for this particular example with these numbers

so it turns out that this is true in general

so let me quickly present that result

so let us say that we have two circles here which which touch at this point and let us say these are the centers of this two circles and they are touching at this point let us say that the equation of the

so this is the first circle

so the equation of the first

circle is represent the first circle by $S_1 = 0$

so this is the equation of the first circle sorry

so i will use something else here i will say O_1 and O_2 for the centers because i will be using C_1 and C_2 in the equation for these two circles

so the centers are O_1 and O_2

so this is the first circle $S_1 = 0$ this is the second circle $S_2 = 0$ this is the equation of the first circle and this is the equation for the second circle

so now what is asked to us is to find out the equation of the transverse common tangent which i have drawn in green

so obviously the the centres of these two circles are (g_1, f_1) for the first circle with center O_1 and the center O_2 for the second circle is (g_2, f_2) and this is the straight line joining the two centers let the coordinates of this point of contact be (x, y) then it is clear that this (x, y) satisfies both these equations now since it satisfies both these equations it must also satisfy the difference of these equations

so if i take the difference what i get is $2(g_1 - g_2)x + 2(f_1 - f_2)y + 2(c_1^2 - c_2^2) = 0$

two into $f_1 - f_2$ into $y + c_1 - c_2 = 0$

so the coordinates of this point of contact x, y satisfies this equation which is actually this is actually a straight line equation this is basically the equation of some straight line we will show that this equation is nothing but the equation of the transverse common tangent

so one thing that we know is that this point of contact x, y lies on the transverse common tangent and this x, y satisfies this equation but what is need to be checked is whether the slope of this equation if this equation has to be the equation of this tangent the slope of this equation must be equal to the slope of this tangent now the slope of the line joining the two centers is equal to $\frac{f_1 - f_2}{g_1 - g_2}$ because this is as follows from the coordinates of the two centers

so this is the slope of the line joining the centers of the two circle now because the tangent is at 90° these the product of the slope of the tangent with this slope must be minus one that is and therefore the slope of the tangent is $-\frac{g_1 - g_2}{f_1 - f_2}$ now if we go back and see this line equation here it is clear that the slope of this line is also equal to $-\frac{g_1 - g_2}{f_1 - f_2}$ and this shows that the equation of the common tangent is nothing but this equation which is basically the difference of the equations of the two circles

so let us take another problem related to common tangents between two circles so in this next problem it is given that the centers of two circles c_1 and c_2 each of unit radius are at a distance of six units from each other let p be the midpoint of the line segment joining the center c_1 and c_2 and let c be a circle

so joining the centers of now let c be another circle touching these two circles externally but it is touching in such a way that a common tangent to c_1 and c passes through passing through p

so this common tangent is also a common tangent to c_2 and c

so what is said in this question is that we have two circles of unit radius c_1 and c_2 as drawn here let us say o_1 and o_2 are the centers and the distance between the two centers is six units the midpoint is denoted midpoint of this line segment $o_1 o_2$ is denoted by the point p and then it is said that there is another circle c which touches both this circle c_1 and c_2

so let this be circle c and it touches c_1 at this point and c_2 at this point and lets say that the let us say this is the center o of this big circle u circle c

so if we connect these centers by a straight line then this straight line will obviously pass through this point of contact or the point where these two circles touch similarly the straight line joining o_1 and o will also pass through this point here where the two circles c_1 and c touch each other and then it is said that there is a common tangent to c_1 and c which passes through p

so it cannot be this tangent here because there is obviously one tangent like this this is like a direct common tangent to c_1 and c but this obviously does not pass through p

so the other case could be that we have a tangent like this

so this red line red straight line is a common tangent to both c_1 and c

so this is a direct common tangent to c_1 and c which passes through p and further this is also a tangent to this this is also a common tangent between c_2 and c

so this is c_1 and this is c_2 and it was said in the question that the same common tangent between c_1 and c which passes through p must also be a common tangent between c_2 and c

so that is what we are seeing here

so since this red straight line is a common tangent between c_1 and c_2 this angle must be ninety degrees these two angles must be ninety degrees

so let the radius of the circle c_1 which is what we have to find out be r and this ah this is of course one unit and this radius here is also one unit

so we look at the triangle $o_1 o_2 o$ the sides two sides are of the same length $r + 1$ equal to $r + 1$

so this side is also $r + 1$ this side is also $r + 1$ then therefore this is an isosceles triangle hence if we and since p is the midpoint of this side $o_1 o_2$ if we connect $p o_1$ it is correct it is clear that the angle $o_2 p o_1$ will be 90° degrees now we know that since p is the midpoint of $o_1 o_2$ then the length of $o_1 o_2$ is six units it is clear that $o_2 p$ is equal to 3 units let this point of contact be denoted by s now we see that this triangle $o_2 p s$ is a right angle triangle and therefore this length $p s$ will be equal to

from the pythagoras theorem nine which is the square of this hypotenuse will be equal to the square of $p s$ plus one and therefore the square of $p s$ is eight and hence $p s$ is and hence $p s$ is square root of eight units now what we also see is that because this red straight line is a tangent is a common tangent between c_1 and c_2 the triangle $p s o_1$ is also a right angle triangle and in

this triangle $p s o_1$ we have one side of length r the other side $p s$ is square root of eight and therefore the hypotenuse $o_1 p$ will be equal to again by the pythagoras theorem square root of $r^2 + 8$ now going back to the triangle sorry going ah then considering the right angle triangle $o_1 p o_2$ which is this right angle triangle $o_1 p o_2$ we see that $o_2 o_1$ square is equal to because this is the hypotenuse $o_2 o_1$ square plus $o_1 p$ square therefore now $o_2 o_1$ this is $r + 1$

so $o_2 o_1$ square is $r^2 + 1$ whole square which is equal to $o_2 o_1$ whole square plus $o_1 p$ whole square $o_2 o_1$ is three units

so square is nine plus $o_1 p$ square is from here $r^2 + 8$ that is and therefore r is eight units

so the radius of this circle c_1 is eight units let us take the third question

so in this third question we have two circles each of radius five units and they touch each other at this point one comma two the equation of their transverse common tangent is given to be $4x + 3y = 10$ and the question is asking us to find the equation of both the circles

so the situation is like this we have two circles which have the same radius which is five units and they touch at this point the coordinates of this point is one comma two let this be the straight line joining the two centers and the transverse common tangent which is this blue straight line has equation $4x + 3y = 10$

so one can check that ah this point one comma two satisfies this equation because obviously the point where these two circles touch has to lie on the transverse common tangent between these two circles

so we already know the radius of these two circles if we can somehow find the coordinates of these two centers then writing down the equation of these two circles will be easy now we know that the tangent makes an angle of 90° degrees with this straight line joining the two centers the slope of this tangent is minus four by three

so slope is minus four by three from this it follows that the slope of the straight line joining the centers must be three by four because the product of the slope of these two lines perpendicular to each other has to be minus one and the slope of this tangent is given from here we see that the slope is minus four by three and since the product of the slope of this tangent and this line joining the centers has to be minus one it follows that the slope of this line

joining the centers must be three by four and therefore the equation of this ah now let us say that the coordinates of the first center here is a comma b of the first circle

so then it follows that this is this is the slope of this line segment is three by four it follows that $b - 2$ divided by $a - 1$ equals three by four

so from here we can say that $b - 2$ is three by four times $a - 1$ one further this length between $0, 1$ and the point of contact here which is $1, 2$ this is nothing but the radius of the circle which is five units and therefore the squared radius which is twenty five will be equal to $b - 2$ whole square plus $a - 1$ whole square but $b - 2$ whole square can be written in terms of $a - 1$ whole square using this equation

so this is equal to nine by sixteen into $a - 1$ whole square plus $a - 1$ whole square

so if we solve this equation we get $a - 1$ whole square equals sixteen which implies that a is equal to one plus minus four

so therefore we have two values for the x coordinate

so in fact these two values will correspond to the x coordinates of both the centers because the second center the center of the second circle is also at a distance of five units from this point of contact

so the coordinates of the the center of the second circle must also satisfy this equation and therefore it follows that the coordinates the x coordinates of these two circles are

so for the first circle the the x coordinate will be

so let us say one minus four which is minus three and the x coordinate of the second circle will be one plus four which is five and now if the x coordinate is minus three which is a then it follows that the y coordinate which is b we can use this equation will be

so $b - 2$ will be 3 by 4 into this goes up here three by four into minus four which is minus three and therefore b will be equal to minus one for this point and for this point it will be five

so now we have the coordinates of the centers of these two circles and we know the radius to be five units and therefore the equation is easy now

so the equation of the first circle will be $x - 3$ whole square plus $y - 1$ whole square is equal to square of the radius which is twenty five and similarly we can write down the equation of the second circle next we consider another little problem or another little topic of finding the common cord

so next we discuss this topic of finding the common chord of two intersecting circles

so the situation here is that we have two intersecting circles like this

so when we have two intersecting circles we have two points where they will be intersecting each other and if we join these two points by a straight line this line segment is a chord for the first circle as well as for the second circle and that is why this chord is called a common chord of these two intersecting circles now suppose that the equation of the these two circles

so the this is S_1 this is S_2

so let's say the equation of the two circles are as follows

so we are given the two equations of these two circles how do we find the equation of this common chord now suppose that let us say that these two the coordinates of these two points of intersection

so let us say that the coordinates of these two points are for this point it is a, b for this point let us say it is c, d

so both these points a, b and c, d

so a, b will also satisfy these two equations c, d will also satisfy

these two equations now because a comma b satisfies these two equations it means that a comma b must also satisfy the difference of these two equations

so when we take the difference of these two equations we get this equation now this equation is the equation of some straight line now clearly a comma b must satisfy this equation because a comma b satisfies both these equations similarly c comma d which is the which is the other point of intersection of these two circles will also satisfy this equation now with that what that means is that both these two points essentially lie on the straight line and since the straight line segment between any two points is unique it follows that this straight line segment or this common chord has the equation which is nothing but this equation

so this is the equation of the common chord of the intersecting circles the length of this common chord is not very difficult to find and we can quickly derive or see how to find it

so let us say that o_1 and o_2 are the center of these two circles

so let us say that this point of contact is p or this point of intersection is p and this point of intersection is q this is the straight line joining the two centers now it is not very difficult to show that this common chord and this straight line now joining the centers will be at will be perpendicular to each other they will be perpendicular to each other let us say this point here is m

so since ah this line joining the center is perpendicular to this chord p q what we also see is that this triangle one q p is an isosceles triangle this is an isosceles triangle because $o_1 p$ and $o_1 q$ are nothing but the radius of this circle r_1 and because the series triangle and because $o_1 m p$ is ninety degrees it follows that m is the midpoint of this chord p q and since m is the midpoint point of this chord p q the length of the chord p q will be equal to two times the length of pm now finding pm is finding pm will not be very difficult because if we see this triangle one p m it is a right angle triangle where $o_1 p$ whole square is $o_1 m$ whole square plus p m whole square and therefore p m is square root of now $o_1 p$ whole square is nothing but the squared radius of the first circle which is let us say r_1^2

so from this equation we have p m equal to square root of $o_1 p$ whole square which is r_1^2 minus $o_1 m$ whole square

so now how we just need to find $o_1 m$ if we find $o_1 m$ then we can find p n but finding $o_1 m$ is very easy because $o_1 m$ is nothing but the perpendicular distance or the length of the perpendicular from this center o_1 whose coordinates are known

so since we are given the equations of the circles the coordinate of o_1 is known to be minus g one comma minus f one

so this coordinate is exactly known to us because the equations of the circles are known further we also know the equation of this chord which is this

so then this $o_1 m$ is nothing but the length of the perpendicular from this known point to this known straight line p q

so it is nothing but the distance between this perpendicular distance of this point from this straight line which can be easily found out and that is our $o_1 m$ and then from there since we already know the radius of this first circle the radius of the first circle will be known from this equation here which is given to us

so then we can find p m and then we just have to multiply it by two and then p q the length of the chord is just twice of p with this we come to an end to this lecture in the next lecture we will start a new topic and we will discuss how to find the angle of intersection between two intersecting circles we will find the conditions under which this angle of intersection is ninety degrees and will also ah define something known as the radical axis between two circles thank you

you

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