welcome to lecture seven on circles

so in the last lecture we had finished the derivation of the equation of the direct common tangents to two given circles for the first case where the circles were not intersecting each other neither were they touching each other for the same case we now resume with deriving the equation of the transverse common tangents to two given circles

so let these be the two given circles with centers c one and c two

so c one is the center of the first circle having coordinates x one y one c 2 is the center of the second circle with coordinates x 2 y 2 and let c 1 c 2 be the line joining the two centers if we recollect the transverse common tangent is a tangent which is common to both the circles but it is such that the circles lie on the opposite sides of the tangent

so this is a this is one of the transverse common tangents because if you look at the tangent then this circle lies on this side of the tangent and the other circle lies on the other side of the tangent and let us say that this is the point p which is the point of intersection of the straight line joining the two centers with the transverse common tangent and let this point p have coordinates gamma and delta gamma and delta let this point be a and we will say that this point here is point b and let us join c one two a and c two to b then clearly this this angle and this angle are ninety degrees and then as before it is quite clear that this angle and this angle are equal

so let us see the two triangles p a c one and the other triangle is p b c two and what we will see that the two triangles are similar to each other because all the three angles of the these two triangles are the same because first of all one of the angles is 90 degree and this angle and this angle are also equal and therefore the third angle has to be equal this and this will also be equal

so given that these two triangles are similar we can now write down the similarity ratios for these two triangles now this distance is r one and this distance e to b is r two

so from the similarity ratios what we get is that p c one divided by p c two p c one divided by p c two has to be equal to r one divided by r2 and if we work it out

so given this fact what it means basically is that the point of intersection of the common tangent with the line joining the centers of the two circles

so this point of intersection divides the line joining the two centers in the ratio of the radius of the circles

so that is what this equation tells us and this division is internal unlike in the case of the direct common tangent where the point of intersection was dividing this straight line joining the centers externally in the ratio of the radii

so here the division is internal now starting from here it is very easy just like what we did earlier to find the coordinates of this point of intersection here

so and that is left as an exercise for the students

so one can show that the x coordinate gamma of the point p is given by r one x two plus r two x one divided by r one plus r two and the y coordinate is given by r one y two plus r two y one divided by r one plus r now ah given that given this ah value of the coordinate of this point of intersection of the transverse common tangent with the line joining the centers of the two circles we can write the equation of ah we can write the equation of the transverse common tangent

so let us say that the slope of this transverse common tangent is m and we know that it passes through this point p with coordinates gamma and delta gamma and delta and therefore we can say that for any point x y or any point x comma y on this transverse common tangent the coordinates x and y of any point on the

transverse common tangent has to satisfy this equation which is y minus delta is equal to m times x minus gamma

so this is the straight line equation for this transverse common tangent but then although we know the we have been able to express delta and gamma in terms of the radius of the circles and the coordinates of the centers of the two circles but the value of m is still unknown and that is what has to be found out

so just like what we did ah yes what we did in the in the previous lecture what we observed from here is that the slope m has to be such that the minimum distance of this transverse common tangent which is this line here

so the minimum distance of this straight line from the center c one of the first circle must be r one and similarly the minimum distance of the same straight line from the center c two of the second circle must be r two and a little bit of calculation will show that ah both these things are one and the same

so we will again get two equations

so again if we go back to one of the slides in the previous lecture we have derived the formula for the minimum distance of a given point x naught y naught from a given straight line

so this straight line is having slope m and it passes through a given point alpha beta then in that case the square distance the square minimum distance of this point from this straight line is given by this expression

so we will again use that expression

so the only thing is that in our case the point ah from where we have to find the minimum distance is the center of the first circle and the straight line to which we have to find the minimum distance to the straight line is actually the transverse common tangent which is known to pass through this point p having coordinates gamma comma delta and this ah transverse common tangent has a slope m and therefore this minimum distance which must be r one because otherwise this line will not be the transverse common tangent to the first circle and and therefore ah from the slide in the previous lecture the right hand side here gives us the squared minimum distance

so we just have to replace x naught and y naught with x one and y one and alpha and beta with gamma and delta

so what we get is the square distance is m into x one minus gamma minus m times x one minus gamma minus y one minus delta square upon one plus m square equals r one square and then similarly we get a similar equation for the second circle also but then just as we had shown yes just as we had shown in the previous lecture even even for this case we can show that the other equation is nothing but the same as this equation

so the other equation that we will be getting is ah for the second circle which is this equation this equal to r two square but then it can be shown that these two are nothing but one and the same and therefore will proceed only with one of the equations and when we solve this equation we will again get a quadratic equation an equation which is quadratic in m which means that there will be two roots real both will be real valued roots and therefore will get two different equations for the transverse common tangent

so if we ah

so that is what we had gotten ah in the previous lecture also

so the roots let m one comma m two be the two roots of let me say this equation to be three of the quadratic equation resulting from equation three and then the equation of the true trans was common tangents is

so the two equations will be one will be y minus delta is equal to m one times x minus gamma and the other equation will be y minus delta is equal to m two times x minus gamma and also interestingly

so let me draw the second transverse common tangent here

so the second transverse common tangent will also pass through p and that is clear because if we if let this be the first if we see here the point p which is gamma comma delta will it can be seen that it lies on this straight line as well as this straight line and therefore the point p lies on both the tangents

so the next case is when the two circles touch each other touch each other externally say

so they do not intersect but they touch each other at one point

so so let us remember what had happened in the previous case in the previous case the circles were neither touching nor intersecting and then we had two direct common tangents and two transverse tangents as well this was the point of intersection of the transverse time common tangents with the line joining the centers of the circle now what if we start moving this circle towards the first circle along the same this line joining the centers then it is expected that for example if if this smaller circle with center c2 the center c2 moves to let us say this point here

so then we have the circle c two the second circle to be like this and let us say that this is the new point c two prime

so in that case we see that ah the transverse common tangents become like this and then this is the point of intersection which is still on the line joining will still lie on the line joining the centers of the circle but we start seeing that the two transverse common tangents the the angle between them which is this angle

so earlier we had this angle and now the angle has reduced and as we move further let us say then what we expect is the moment this smaller circle touches the first circle we expect that these two tangents will probably become the same one they will become one single transverse common tangent

so ah to to see that ah one could actually go back if we go back to the first case and especially when we were deriving the equation of the two transverse common tangents we ah we had this particular quadratic equation and we said that there will be two equation two ah roots ah m one and m two for the slope but then what we can see is that from here if as we will derive it now that as the when the the second circle touches the first circle in that case we will have two equal roots here

so this quadratic equation is going to have two equal roots which means that there will only be one transverse common tangent

so so to to see that

so obviously when if we if we recollect the coordinates of the point p was gamma comma delta where gamma and delta were given by these two equations and if we if we if we open up this particular quadratic equation a little bit of simplification gives us this quadratic equation

so what we get is m square into r 1 square minus x 1 minus gamma whole square plus 2 m into x 1 minus gamma into y 1 minus delta plus r 1 square minus y 1 minus delta square this equal to zero

so this is the quadratic equation that we will get from here we just have to take this here and then just do a shuffling of the terms and this is what you get

so this point p

so now when this let us see the condition under which this quadratic equation is going to have equal roots

so it will have equal roots if and only if the discriminant is 0 and the discriminant is 4 into x 1 minus gamma whole square into y 1 y 1 minus delta whole square minus 4 times r 1 square minus x one minus gamma whole square times r one square minus y one minus delta whole square and we need this to be zero

so if if the coordinates gamma and delta of the point of intersection of the tangents transverse common tangents satisfy this equation only then will it happen that will have only one transverse common tangent but if we if we simplify this further because the radius of the first circle is not zero what we will get is that we will get r 1 square

so this condition is the same as the condition that r 1 square is x 1 minus gamma whole square plus y1 minus delta whole square but what this is telling is that and if you remember this distance this distance is nothing but the distance between the point

so this was the point p

so this is the point p over here

so this distance is nothing but the distance between p and the center c one

so square of that

so p c one square

so essentially what this implies is that will only have one transverse common tangent only if if and only if if and only if p c one is r one which means that the distance between the point of intersection of these two of well now that we only have one tangent in this case when the

so so when we have only one common tangent the point of intersection of that tangent with the line joining the two circles that point is such that the distance of that point from the center of the first circle is r one what this basically means is that this point is actually on the it's on the first circle it's on the circumference of the first circle

so so essentially what we have shown is that the we will have only one transverse common tangent only in the case where the point of intersection of that common tangent with the uh the

so the the point of intersection of that particular common tangent with the straight line joining the centers of the two circles

so that point of intersection p is lying exactly on the circumference of the circle

so p c one is equal to r one and this is the transverse common tangent

so there is only one transverse common tangent when we have only one root or basically both the roots are equal

so under that situation we have only one transverse common tangent and in that case we saw that the point of intersection of this only this single transverse common tangent with the line joining the center

so that particular point of intersection p will lie on the circumference of the circle now since this is a common tangent this is a common tangent to both the circles it means that this is also this line is also a tangent to the second circle and further we know that this point p lies on the straight line joining the two centers

so if we just produce this line further which means that this angle is also 90 degrees the center of the second circle has to be somewhere on this line

so we know that the center of the second circle has to lie on this line which we have produ which basically we have joint c one with p and we have extended it further and the second of the second circle also must have its center on this line plus we know that this same straight line here which is a tangent to the first circle is also a tangent to the second circle

so therefore the the shortest or the minimum distance of this straight line from the center of the second circle must be the perpendicular from the center of the second circle to the straight line but the perpendicular is just this line this part of the line segment which also passes through p and therefore it is clear that p is the point p basically must also lie on the circumference of the second circle

so we have a situation like this and now we know precisely that this point p lies on both the circles and it also lies on the transverse common tangent right and therefore it is clear that both the circles are now because this point is lying on both the circles it is clear that this point common to both the circles and therefore both the circles are actually touching at only this point and they are not touching they are not intersecting because if they had if if there was an intersection something like this

so if we have this situation then there is no we cant have any transverse common tangent

so this situation will basically happen when the quadratic equation that we have here has no real roots

so the first case was when it had two real roots

so when it had two real roots it was the first case where the two circles were neither touching nor intersecting the second scenario is what we were doing just now where this quadratic equation just has a single root

so when that happens both these two circles will be exactly touching exactly touching each other at a single point and they will just have one transverse common tangent then further if this circle centre of the circle c 2 moves towards c 1 which could be this case in this case what will happen is that this particular quadratic equation will have no real solutions for m and that is why there will be no transverse common tangent in this case also now in this case which is the second case where there is one transverse common tangent it is very easy to find the equation of this particular tangent

so that will be

so we just have to again ah

so basically we will just we wont have two tangents we will just have one tangent and the equation will be of this single transverse common tangent will be y minus delta is equal to m into x minus gamma where m is the value of the the equal roots of that quadratic equation in m

so this is only this single transverse common tangent and of course how how do we suppose if we are given two circles let us say and if we are only given the equation of those two circles

so we are only given the equation of two circles and then we are asked to find out whether is it condition one that is happening or is it condition two that is happening

so for condition one we had we had said that will find the distance between the two circles within the center of the

so this was

so this was condition one here

so this is this is case two and this was case one where the two circles were neither touching nor intersecting each other and for this case one

so for case one we had said that if we are given the equation of the two circles

so from the equation we can find the coordinates of the center we can also find the value of the radius of the two circles from the general equation of these two circles and then what we can do is we can find the distance between the two centers and if the distance between the two centers happens to be strictly greater than the sum of the radii or the sum of the radius or two circles if this happens then it is clear that the two circles are neither touching nor intersecting each other but if it

so happens that the distance between the two circles which in case two this is exactly equal to r one plus r two and and i mean given the two equation of circles we can easily find the coordinates of the center and therefore we can easily find this distance

so we can find this left hand side and of course from the equation of the circles we know their radius we can add the radius and if these two are exactly equal then we know that we are in case two where we have only one transverse common tangent but of course in case two we will still have two direct common tangents and finding the equation in of the direct quantum tangents will be the same as that in case one

so now we can take the third case which we have already discussed a little bit
so this third case is where the circles happen to intersect each other
so if the circles intersect each other
so firstly how do we find out that they are actually intersecting each other
so we will again given the two equations of these two circles we will find the distance c one c two the distance between the two centers and we will also find the radius of the circles
so if c one c two is less than r one plus r two then it is clear that it is neither case one nor case two but then we could have two possibilities if this happens if this happens this is one possibility the other possibility could be something like this
so c one is here c two is there
so this is the smaller circle and this is the bigger circle the bigger circle has center c one the smaller circle has center c two or we could even have this kind of a situation where the the smaller circle is touching the bigger circle in from internally from inside
so how do we distinguish this case from these other cases what we can say is that if the distance between the circles is greater than modulus of r one minus r two then it has to be this case because what will happen is that
so the how do we have we have got to this third case by just you know ah basically moving this smaller circle along this line
so earlier the smaller circle was somewhere here
so this was case one where they were neither intersecting not touching then the smaller circle came and over somewhere over here and at this
so this was case two this was case one
so in case two it was exactly touching the bigger circle at just one point and then if you move this circle from here to here and then further ahead then of course we are in case three where they are intersecting and then if we even move it further we will arrive at this case
so even if we even move it further then we will actually arrive at this case where the the smaller circle is something like this
so in this case the smaller circle is actually touching the bigger circle in blue from inside
so it is touching it internally but in this case if you see this is r one this is r two
so this is basically this case
so if we if we take this case this this is r 1 and this is r 2 or if we can draw it separately
so this is a smaller circle center c one c two
so this is r one and this is r two
so therefore the distance between this two centers is r one minus r two and we just take a modulus because of course here we are assuming r one to be larger than r 2 but it could be the other way around
so that is why we have to take modulus
so so if if it if the if the smaller circle comes you know keeps on coming closer the center c2 keeps on coming closer till ah this case happens
so as long as this at the distance when this case happens we have c1 c2 equal to mod of r one minus r two if the distance c one c two is greater than this if

it is greater than this value then it is clear that the circle is not touching internally this are the smaller circle is the smaller circle is something like this

so so this is the case that we have now where this circle

so in this case where the distance between the two centers is first of all less than r one plus r two but it is greater than the absolute difference between r one and r two

so in that case what we will have is that the two circles will be intersecting which is with each other

so it will be something like this

so they will be intersecting at exactly two points and of course for this case then what will happen is that there will be no transverse common tangents because what will happen is that the both the roots will become non real but we will still have two direct common tangents whose equation can be found again using the methods used in case one and then of course we have the case where the two circles are going to touch each other internally

so if we move the center of the the smaller circle even closer to ah c one then

so in the previous case in

so so earlier we had this case 2 where the center of the smaller circle was such that the distance between the centers was r 1 plus r 2 and then we moved this circle a little closer along the same line joining the centers

so the circle the center came to this point and the the circle was then something like this this smaller circle was something like this and this became the center

so this is the case this is case three in this case we have already seen that they are intersecting and the two circles intersect at two distinct points and then if we further move this circuit the center of this circle c2 towards c1 along the same line in such a manner that they touch the second circle touches the first circle internally

so what we mean by that is the second circle basically has center c2 in such a way

so this is center c2 in such a way that this red circle smaller circle and the bigger blue circle touch exactly at this point

so this point is p and of course this will happen only when when c 1 c 2 is equal to the absolute difference between r one and r two because this is r one n this is r two at the center of the smaller circle comes closer and closer to the center of the first circle till ah this condition is satisfied

so the moment this condition is satisfied it is we can clearly see that the second circle touches the first bigger circle internally at this single point of contact p and in that case there is one common tangent there is only one common tangent and this is a direct common tangent

so this is

so in this case there will be no transverse common tangents there will be just one there will be only a unique direct common tangent and whose equation is not very difficult to find

so basically we will again have to use the the analysis for the direct common tangent that we saw in the previous lecture for the first case if you remember the first case

so if you remember this slide

so this slide was for finding the slope m of the direct common tangent for case one where the two circles were neither intersecting nor were they touching each other in that case we had got a equation that was quadratic in m when we said there will be two roots but in this fourth case in this fourth case when the distance between the circles is absolute difference between the radius in that

case what will actually happen is that this quadratic equation will only have one root only one single real root there will be only one single real root which basically means that there will be only one direct common tangent and whose equation can be easily found out

so so the the coordinates of this point will be alpha and beta

so the value of alpha is

so this is the value of alpha and beta can is will be correspondingly similar

so so alpha will be r 1 x 2 minus r 2 x 1 by r 1 minus r 2 and beta will be r one y two minus r two y one upon r one minus r two and then once we know ah know this ah coordinates of this point the equation of this particular direct common tangent will be y minus beta is equal to m into x minus alpha because we know that this point will lie on the direct common tangent and the value of m can be will be will get the value of n from m will get the value of m by solving this quadratic equation which will have equal roots

so both the roots will be real and equal for this case four where the two circles are touching each other internally and of course in this case there will be no transverse common tangents

so the number of transverse common tinies will be zero and then further as the center of the smaller circle lets say of this other circle moves even closer to c one and of course the last case is when the center of the second circle moves

so close to the center of the first circle along the same line along the same line that we had in the earlier cases

so we were moving the center of the second circle on this line closer and closer to c one

so we had

so this was case two when the circles were touching each other and then this was case three when they were intersecting each other and then we had case four when the two circles were touching each other internally and then if we move the center even further

so then we can have a case something like this where the c two the center of the second circle is here this is the second circle but then the two centers are

so close that c one c two is less than the absolute difference between the radii between the radius

so in this case as we can see the two circles neither intersect neither do they touch each other and the second circle is completely inside the first circle

so this is the fifth case

so in this fifth case it is clear that there will be no direct common tangent and there will be no transverse common tangents

so let us solve some problems to ah for getting used to finding equations of common tangents

so in this question we are asked to find the number of common tangents to the circles x square plus y square equal to four and the other circle is x square plus y square minus six x minus eight y equal to twenty four

so this first circle the coordinates of the center is at the origin radius is two units for the second circle the center is at three comma four and the radius is seven units the distance between the two centers is five units and we see that this is equal to modulus of r one minus r two which is five

so this is precisely case four that we were discussing just a few minutes back

so when the distance between the centers is equal to the absolute difference between the radius

so this basically means that the two circles are touching each other internally and therefore there is only one direct common tangent there are no transverse common tangents

so the answer is there is only one direct common tangent

so let us draw this try to find out the equation of the direct common tangent

so this is the coordinate axis this is the first circle this is the circle c2 having center c two and radius r two equal two the other circle has center c one and radius seven which i am just drawing here but obviously we cannot draw the entire circle because it has a very this radius is seven units and as you can see these two circles touch internally at this point p

so the

so they have only one direct common tangent further this coordinates of the point of contact we have already seen the expression

so using that we will get alpha equal to r one seven times x two

so this is x 2 y 2 this is x 1 y 1

so x 2 y 2 is are both of them 0

so both x 2 and y 2 are 0 7 times x two minus r two times x one

so two times three by r two minus sorry r one times x two minus r two times x one by r one minus r two

so this comes out to be minus six by five and the y coordinate of this point will be r one times y two minus r two times y one by r one minus r two minus eight by five now once we know the coordinates and we also know that

so let this be the straight line

so the straight line joining the center c 1 and c 2 when produced forward when produced further will also meet this point p which is the point of contact of the two circles and therefore and further that this tangent will make a 90 degrees with this straight line and therefore is it it is easy to find the slope of this direct common tangent because this this this line is at 90 degrees to this line joining c1 and c2

so this is the c 1 c 2 line from c 1 to c 2 and then if you produce it further it will meet the tangent at this point p now the slope of this line is four by three because four minus zero divided by three minus zero

so the slope is four by three and therefore the slope of this line because we know that if there are two perpendicular lines then the product of the slopes is minus one and therefore the slope of this line which is perpendicular to this c one c two line has to be minus three by four and then it is very easy to write down the equation because it will just be y minus beta which is is equal to slope multiplied by x minus alpha

so the equation is y plus 8 by 5 is equal to minus three by four times x plus six by five

so with that we come to an end to this lecture we will take up some more problems on finding the equation of this common tangents in the next lecture thank you you