welcome to the sixth lecture on circles in this lecture we will be discussing the derivation of the common tangents to two circles ah but before we start that let us finish one of the topics that we could not cover in the last lecture

so this is regarding the definition of something which is known as the director's circle to a given circle

so suppose that we have a circle with some center here at o and some radius

so given a circle

so this circle is given to us and then let us think about the locus of all those points which lie at the intersection of two tangents to this circle which intersected 90 degrees

so for example let us say a tangent at this point and then we have to consider another tangent to this circle which is going to be perpendicular to this first tangent

so let us say probably somewhere here we have another tangent here

so we will make a perpendicular tie line to this normal

so this is the other tangent and let us say that these two tangents meet at 90 degrees then we are interested in the locus of these points of intersection

so this is the point of intersection of these two tangents to the same given circle but the tangent should be meeting at 90 degree should be perpendicular to each other let this be point p

so locus of all such points and we can check

so locus of all such points is going to form a circle actually because if we have any such point like this point p over here then clearly clearly this angle is 90 degrees because this is the first tangent similarly this angle is also 90 degrees plus we are given that these two tangents are meeting at 90 degrees

so obviously if you look at this quadrilateral o q p s o q p s if you will look at this quadrilateral then three angles of this quadrilateral are at are ninety degrees

so naturally the fourth one has to be also ninety degrees and therefore this quadrilateral can either be a rectangle or a square but then we see that os and oq are both equal to the radius of the first circle here that is given to us and therefore it is clear that oq ps has to be a square

so what that basically means is that this distance is also equal to r and therefore the distance from o to this point p it will be equal to square root of two times r

so any such point p

so we could we can make another two tangents which are perpendicular to each other and even for example we could have another two tangents let us say tangent at this point like this and then we need to have another tangent which is perpendicular to this tangent

so let us say a tangent at this point over here

so these these two tangents meet at 90 degrees and if we do a similar analysis like what we did for this point p let us call this point as c

so if we do a similar analysis then what we will also see is that

so this is again going to be a square of side equal to r and again this distance oc is going to be equal to square root of 2 times r where r is the radius of the given circle

so what we see is that any such point which lies at the intersection of two tangents which are intersecting at ninety degrees

so any such point is going to be at a fixed distance of square root two times r from the center of the given circle and therefore the locus of all such points is another circle because that was what was the that was basically the definition of a circle because each such point which lies at the intersection of two tangents which intersect at 90 degrees

so each such point is going to lie at a fixed distance from the center of the given circle and therefore this circle that we get which is the locus of all such points is called the director circle to the given circle and as we can see the center of the director circle is the same as the center of the given circle

so that is the first observation that the center of the director circle to a given circle is same as the center of the given circle there is one observation and the other observation is that the radius of the director circle is square root of two times the radius of the given circle

so with that we finish this discussion on the director circle next we are going to talk about the common tangents of any two given circles but before we do that there is one little small result that probably would have been covered in one of the previous lectures but we just bring it again here because in our analysis we will be using this result

so the result is like this it basically says that if we have a straight line over here the slope is m and which passes through this point alpha beta

so and then we have another point x naught y naught and the question we are asked is to give an expression of the square distance the minimum square distance of this point x naught y naught from any point on this straight line

so obviously as we know from high school the shortest distance or the smallest distance is basically the perpendicular from this point on to the straight line which is this perpendicular and this quad distance of this perpendicular

so the square distance of this line segment from this point to this straight line is given by this formula

so let us talk about common tangents between two circles

so let us say that there are of course many cases here

so let us draw any two circles here

so this is the case when the two circles are neither touching each other nor are they intersecting each other

so let these be the centers c one c two then we can see that there are actually four tangents

so two of the tangents

so four common tangents

so what what we mean by a common tangent common means that the same straight line is a tangent to both the circles for example let us say this straight line

so this straight line that i have drawn is a tangent to both this first circle

so this straight line is a tangent to the first circle at this point and the same straight line is a tangent to the second circle at this point here

so this straight line is therefore called a common tangent to both these given circles

so we can draw another ah common tangent here like this

so these two tangents are called direct common tangents but apart from these two we will still have two more tangents which are called transverse tangents and they are like this as we can see this red line is a tangent to the first circles at this point and the same on the same red line is a tangent to this other circle at this point

so in the case of a direct common tangent both the circles are on one side of the tangent if you see this

so as we know any straight line divides the surface into two halves one half is on this side of the straight line for example let us take this straight line and the other half is on the other side of the straight line in the case of a direct common tangent both the circles are on one side of the tangent one on one side of the straight line tangent similarly this is

so such a tangent for which both the circles are on one side of it that type of a tangent is called a direct common tangent

so this green one is one of the direct carbon tangent because both the circles are on or below this straight line similarly this other green tangent is also a direct common tangent because both the circles are above the or on one side of the this straight line but in the case of the red tangent the red tangent obviously divides this surface into two parts one is this side the other is this part and in the case of the red tangent we can see that this bigger circle is on this side and the smaller circle is on the opposite side
 so such a tangent for which the two circles are not on the same side of the tangent is called a transverse common tangent
 so this is also a common tangent because this straight line is a tangent to both the circles but then the circles are on opposite sides of the tangent transverse common tangent
 so there will be one more um transverse common tangent which is like this
 so in this first case there will be totally four common tangents two of them direct and the other two are transverse
 so in the next part of this lecture we will see how to derive the equations of this direct common times all the four and also the coordinates of the point of intersection of these common tangents
 so let us start with the derivation of the equation of the direct common tangents for this first case
 so let these be the two circles the center c one and center c two c one has coordinates x one y one c two has coordinates x two y two and let the
 so let the radius of the first this circle with center c one be r one and that of the other circle with center c to be r two
 so how do we suppose if we are given we are i mean its not suppose the circles are not geometrically drawn and what we are given is only lets say the radius of these two circles and the coordinates of these centers of the circles then how do we check whether the case is how do we check whether the case is that of two non intersecting and non touching circles
 so for that it is not very difficult what we realize is that if the distance between if the straight line distance between the two centers
 so if the
 so the distance between the two centers which is actually this expression if this distance between the two centers is greater than the sum of the radius
 so this we can easily check if this condition happens then as we can see if this condition happens then it is clear that because if we lets say we join the centers with a straight line then this distance is r one from here to here and this distance is r two
 so obviously if the two circles are not touching and not intersecting then it is clear that the total distance between the two centers will be r one plus r two plus something more because the two circles are not neither are they touching nor are they intersecting
 so obviously when that happens this has to be true and vice versa also if the distance is more than r one plus r two then that also implies that they do not touch each other or they neither do they intersect
 so let us take this first case where we have two non intersecting and non touching circles
 so lets say these are the two circles
 so one has center c one the others has center c two let the coordinates of c one and c two be x one y one n x two y two let the radius of the first this bigger circle with center c one b r one and that of the smallest circle with center c to be r two
 so naturally r one is greater than r two for this example here now let us consider this common tangent

so let the point of contact of this direct common tangent to both to the first circle b at a point a and the point of contact or the point where this direct common tangent touches the second circle is let that point be b

so this is a this is b clearly these angles are 90 degrees and let us consider the straight line joining the two centers of the circle let us extend that forward

so clearly this straight line is going to intersect the direct common tangent at some point p whose coordinates we denote by alpha comma beta

so our first job is to find the coordinate of this point p alpha comma beta and then we shall find the equation of this direct common tangent

so now this is r one and this is r two now let this distance between the two centers of the circle let me denote it by l one and let the distance between this point p and the centre of the second circle c two be l two we observe that we observe that the triangle p b c two which is this triangle is similar to triangle p a c one and this is because all the three angles of these two triangles

so p b c two and p a c one both these triangles have the same three angles because as you can see one angle is 90 degree this angle is equal to this angle and further it is clear that this angle is also common to both the triangles and therefore because two angles of both these triangles are same the third angle house also has to be the same and because all the three angles are now same for these two triangles it follows that both these triangles are similar and therefore from the similarity ratios and therefore from the similarity ratios it follows that p c one length p c one of the bigger triangle divided by the corresponding side length p c two of the smaller triangle is equal to

so p c one by p c two is r one by r two now p c one is nothing but l one plus l two divided by p c two is l two which is one plus l one by l two and that is r one by r two therefore n one by l two is r one minus r two upon r two which implies that l two is equal to l one into r two divided by r one minus r two and l one is already known to us because we the coordinates of the two centers are given to us

so from here we should be able to find the coordinates of this point p now the equation of

so since this point p is lying on the straight line joining c1 c2 it follows that beta minus y1 divided by alpha minus x1

so this is the slope of this line pc1 the slope of the line pc1 is this and that slope is the same as the slope of the line c 1 c 2 because it is the same line anyways and the slope of the line c 1 and the slope of the line c 1 c 2 is and this in fact is also equal to the slope of the line p c two which is beta minus y two upon alpha minus x 2

so now we will try to find

so we already have got l 2 here in terms of l 1.

now from here what we see is that this distance l two is equal to

so l two square let us say is equal to beta minus y2 whole square plus alpha minus x2 whole square which i will take to the next slide

so in the previous slide we had l 2 square is beta minus y 2 whole square plus alpha minus x 2 whole square which is equal to i will take alpha minus x 2 whole square as common outside this multiplied by one plus beta minus y two whole square by alpha minus x two whole square but beta minus y two divided by alpha minus x two is nothing but the slope of this straight line joining the centers of the two circles which is actually equal to this quantity here

so we can do replace this here by the value

so that implies that l two square is alpha minus x two whole square into one plus y two minus y one whole square upon x 2 minus x 1 whole square but then

from this equation here we already know that l 2 square is l 1 square r 2 square by r one minus r two whole square

so if we use this then we equate this to be equal to l one square r two square by r one minus r two square which is equal to now l one square l one is the distance between the centers of the two circles

so l one square is

so this becomes r two square upon r one minus r two whole square into l one square is y two minus y one whole square plus x two minus x one whole square

so this and this expression are equal and then we see that we can take this as common denominator here and then there will be something which is and then what we

so this left hand side this entire thing can be written as alpha minus x two whole square by x two minus x one whole square into x two minus x one whole square plus y two minus y one whole square but this entire thing is equal to this which is which is multiplied by this expression is the same as this

so and then of course we see that this and this are the same

so what we end up getting is that alpha minus x two is equal to r two by r one minus r two into x two minus x one and if we simplify that further we get the

so value of alpha to be x two plus r two into x two minus x one divided by r one minus r two which is r one x two minus r two x one divided by r one minus r two

so that is what is alpha remember alpha was the x coordinate of this point of intersection of the straight line joining the centers of the circle with this direct common tangent similarly we can find beta and that is easy because we can use this equality over here because since alpha is known now we can easily find beta and little manipulation gives us beta equal to r 1 y 2 minus r 2 y 1 divided by r one minus r two

so now we have the coordinates of this of this point p but our ultimate goal was to find the equation of this the equation of this direct common tangent

so how do we get that

so of course one thing is that we know that this tangent lies on this point whose coordinates are known

so therefore if there is any point x comma y on this direct common tangent then we can say that y minus beta upon x minus alpha

so y minus beta upon x minus alpha is going to be the slope of this direct common tangent and let that slope be equal to m

so now if we know the slope of this direct common tangent then we have finished finding the equation of this direct common tangent because then this is the equation of the direct common tangent but right now m is not known to us

so how do we find m alpha and beta are known here but m is not known what we see is that and this is where the very first result that we saw on the first few slides which was about the shortest distance of a point from a given straight line

so this is where this result is going to be very useful because what we see here is that the shortest distance of this tangent from the centers of the first circle is r one and from the center of the other circle is r two

so this value of m has to be such that the shortest distance of this because if the m is different then the shortest distance might not be r one and r two from these two circles but we know that since this is a direct common tangent if you see this angle here is ninety degrees

so this distance r one is indeed the shortest distance bit of this center c one from this direct common tangent and similarly the shortest distance between the center c two of the second circle and this direct common tangent is r two but the m has to be such that because it is clear if we change the slope like this

then the distances will not be r 1 and r 2 i mean i could i could have for example if i if i had the slope to be something else then i would end up having a straight line like this which although which passes through alpha beta
 so this black line also passes through alpha beta but then since the shortest distance of this black line from the circle is not equal to because this this particular black line is not even touching these two circles which means that the shortest distance from the center of these two circles to this black line will obviously not be r 1 and r 2 because if it is r 1 and r 2 it is clear that this black line has to be a direct it has to be a common tangent to both the circles
 so we will use that to get some equation in terms of m and then try to solve for this m
 so we will use this fact
 so if we go back again in this figure then the first the first equation that m should satisfy is that ah this particular straight line should have a distance equal to r one from the point x one y one
 so we are interested in
 so if we come back here we basically have a point which was the center of the circle c one with coordinates x one y one and there was this direct common tangent whose equation was whose line equation was y minus beta or basically this particular tangent was passing through a point alpha beta then the shortest distance the expression of the shortest distance from c one to this common tangent passing through alpha beta has to be r one
 so now the shortest distance can be calculated using this formula we know we have already said that the slope is m
 so the shortest distance will be equal to m times x one minus alpha minus y one minus beta whole square by one plus m square
 so this is the square distance
 so this has to be equal to r one square
 so this is the first condition that this m must satisfy which is that
 so this expression is for the shortest distance between this point x1 y1 which is the center of the first circle and this direct common tangent or rather this straight line having slope m and passing through a point alpha beta
 so that is what we had we had a straight line having slope m and passing through this point alpha beta
 so we have calculated this distance and the expression for the distance is this left hand side but that must be equal to r one actually
 so this is the square distance
 so since ah for this straight line to be a to be a direct common tangent the shortest distance has to be equal to r one and therefore this expression for the squared distance has to be equal to r one square and we will get a similar equation for the second circle because the same straight line is also a tangent to the second circle
 so that is how we end up getting these two equations
 so this is for the first circle and this is for the second circle now if we try to simplify this equation actually both these equations are one and the same i mean because and that one has to observe it a little keenly to see that point because because if we try to rewrite this we can write it as x 2 minus alpha whole square into
 so i am rewriting the second equation m minus y two minus beta by x two minus alpha whole square is equal to r two square into or rather you can have it in the denominator and this same thing can be written as m minus y two minus beta by x two minus alpha whole square by one plus m square equals r two square by x two minus alpha whole square and

so the same thing we will try to do with the first equation

so we have

so this is ah we have actually

so i will say two prime

so from two we can get two prime very easily similarly from the first equation will get 1 prime just doing it you know just taking out x 1 minus alpha out from here will get m minus y 1 minus beta over x one minus alpha whole square by one plus m square is equal to r one square by x one minus alpha whole square

so this is one prime and two prime was

so this was two prime this equation was two prime now if we ah go back just a few slides back then we know that because

so this is x 1 y 1 this point c 2 is x 2 y 2 and then we have alpha beta

so we we already know that y 1 minus beta

so y1 minus beta upon x1 minus alpha which is this quantity is nothing but the slope of this line joining the two centers of the circle and similarly that and that slope is nothing but y2 minus beta by x2 minus alpha which is this quantity because this is one in the same straight line

so the slope of this line segment in the slope of this line fragment is the same

so therefore these two quantities are essentially the same only and if that is actually also clear from here

so this and this are the same

so therefore what we see is in both these equations one and two the left hand side here is the is the same now what about the right hand side it turns out that the right hand side is also the same because if we recollect then if we if we go back to the similar triangles we see that r one by r two is equal to p c 1 by p c 2.

so if you remember r 1 by r 2 is p c 1 by p c 2 which is equal to which implies that r one square by r two square is p c one square by p c two square now p c one square p c 1 square is

so this is the expression for p c 1 square x 1 minus alpha whole square plus y one minus beta whole square upon p c two square is x two minus alpha whole square plus y two minus beta whole square and this is equal to x 1 minus alpha whole square into 1 plus y 1 minus beta by x 1 minus alpha whole square upon x two minus alpha whole square into one plus y two minus beta upon x two minus alpha square

so y two minus beta square upon x two minus alpha square now we have already seen that this ratio here and this ratio here are the same because they are nothing but the square of the slope of the straight line joining the center of the circle

so these two cancel out what we get is r one square by r two square is equal to this and from this it follows that the right hand side of these two equations are also the same

so we already had seen that the left hand side was same now the right hand side is also same and therefore these two equations are one and the same only and therefore will only we need to only solve only one one of these equations we can take any one it does not matter

so we will only take one of these two equations and solve for m

so let us take the first equation from where in where we had m minus

so by s i will denote the slope of this line joining the

so s is going to be the slope of line joining the centers c1 and c2

so we get m minus s whole square by one plus m square is equal to r one square by x one minus alpha whole square and we already have the expression for alpha

alpha was r one x two minus r two x one by r one minus r two and therefore x one minus alpha is that is x one minus alpha is r one into x one minus x two upon r one minus r

so even we ah put this equation back here we get m minus s whole square by one plus m square equals r one minus r two whole square by x one minus x two whole square and then if we rearrange this what we actually get is a quadratic equation in m which actually means that and in this case there will be two real roots there will be two real roots but then what it means is that there are two different values of the slope which means that there are possibly

so once we solve this will get two solutions will get m equal to m one and m equal to m two

so let us say that this right hand quantity will denote it with k because we already know r one and r two we know x one x two

so let us denote it by k

so then what we have is if we if we

so we have m minus s whole square by 1 plus m square is k and therefore from that it follows that m square minus 2 m s plus s square is k plus k m square and that can be further written as m square into k minus one plus two m s plus k minus s square equals zero

so when we solve this quadratic equation in m we get two solutions m one and m two and therefore we correspondingly get two straight line equations y minus beta is equal to m one into x minus alpha and the other is y minus beta is m 2 into x minus alpha and both of these are valid direct common tangents in fact if we go back then if we recollect our initial discussion we had actually shown that there are actually two in the figure we had shown that there will be actually two direct common tangents for this case for the first case and the slopes of these two are m one and m two

so the the other direct con common tangent will be something like this and this other direct common tangent will also pass through alpha beta and because that is clear from this equation which means that this other common tangent is also going to pass through this point from from the point p

so essentially both the common tangents and the straight lines joining the centers of the circles they are all meeting at this point p and further there is one more it is also easy to see that

so these two direct common tangents meet at this point p which lies on the straight line joining the centers of the circles and this point p divides the straight line joining c1 c2 externally in the ratio of their radii

so what i mean to say here is that

so this is the point of intersection alpha beta this is the straight line joining the two centers and we say that this point p divides this straight line joining the centers externally in the ratio of the radius which means that

so what we are telling here is that since the division is external what it means is that p c one divided by p c two is equal to r one by r two and that is something that we had already we had already mentioned and because i mean this clearly followed from the similarity of these two triangles

so what this means is that this point p where the two direct common tangents meet divides the straight line joining the centers of the circle externally in the ratio r 1 to r 2 r 1 is to r

so p c 1 divided by p c 2 is equal to r 1 divided by r 2 in the next lecture we are going to derive the equation of the transverse common tangents to this circles to both the circles when both the circles do not touch each other neither do they intersect each other thank you you