

welcome to lecture five on circles in the previous lecture we had derived formulas for the equation of the tangent and normal to a circle we had also derived the an expression for the distance of the length of the tangent from a point to a circle

so in this lecture we will take a few problems ah related to tangents ah to a circle just to revise what we had done in the last lecture and then we will define what is meant by the power of a point with respect to a circle thereafter we are going to study the relations relation between two circles to be precise we are going to derive the expression for the common tangents of any two given circles

so let us start with solving ah just two problems

so here is the first question it says that a tangent pt is drawn to the circle $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1)$

so a tangent is drawn to the circle at this point a straight line l perpendicular to the tangent pt is tangent to another circle which is given by this equation and we are asked what what are the possible equations for this straight line l which happens to be perpendicular to the first tangent but the straight line l itself is a tangent to some other circle

so to let us try to explain this through figure

so the first circle was $x^2 + y^2 = 4$

so let us draw the coordinate axis here let us say this is the y axis and this is the x axis we have the origin here

so the first circle is $x^2 + y^2 = 4$

so this particular circle has center at the origin and has radius equal to two

so the first circle c_1 is

so the first circle c_1 is going to pass through all these four point dots that have shown because it has a radius equal to two

so it will appear something like this

so this is our circle c_1 and the other circle c_2 has whose equation is given by $(x-3)^2 + y^2 = 1$

so obviously this circle has a center at uh three comma zero

so which is here and has a radius equal to one

so so this is the other circle c_2

so as you can see it is clear that both these circles are going to touch each other at this point which is two comma zero it is said that a tangent pt is drawn to the first circle at a point p which is square root of three comma one

so the point p is square root of 3 comma 1

so let us see where this point is

so the y coordinate of this point is 1

so and the x coordinate is

so it should be this point because this is the only point this point is in the first quadrant and has a y coordinate equal to 1

so y coordinate equal to 1

so the only point is this

so this is our p and the tangent at p to the first circle c_1 will appear something like this

so this is going to be the tangent pt

so this is our let us say this is d

so this is this red line is the tangent pt to the first circle c_1 and then it is said that a straight line l perpendicular to pt is a tangent to the this other circle

so its just said that the straight line l is perpendicular to pt

so to find ah and then it is of course said that it is also a tangent to this other circle

so how do we do we really need to find the equation of this tangent well the answer is not really that because we are just the only information about $p t$ that is getting used is that this straight line l is perpendicular to $p t$

so what just matters is the slope of this tangent $p t$

so if you look at this tangent $p t$ the slope of this tangent $p t$ is not very difficult to find but then it is said that the straight line l is perpendicular to $p t$

so when it when we say that it is perpendicular to $p t$ what it means is that let us say o is the origin now we know that this ah this line segment $o p$ has to be perpendicular to $p t$

so because this has to be 90° degrees this is because of the property of the tangent itself and then it is they said that the straight line l is also perpendicular to $p t$

so what that means is that the straight line l will be basically parallel to this line segment $o p$

so the straight line l is parallel to $o p$

so therefore for this straight line l this is very easy to find the slope now

so the slope of the straight line l will be simply equal to the slope of $o p$ which is equal to

so so that will be equal to one minus zero divided by square root of three minus zero which is equal to one by square root of three because this point p is actually square root of three comma one and o is the origin

so it is very easy to find the slope and therefore the equation of l will be of the type y is equal to $m x$ plus c

so it will be of the type x divide x multiplied by the slope plus c where c is a constant

so this is the equation of the straight line l and because the question is asking us which of these four possibilities is a valid equation for l such that it is actually this line l has to be a tangent to this to this circle to this smaller circle now if if this if this straight line has to be a tangent what it means is that there is only one point where the straight line must touch the circle c_2

so suppose if there is a point

so suppose that l touches c_2 at some point x comma y

so let us say that l touches c_2 at some point x comma y then it is clear that the coordinates of this point where l and c_2 touch each other the coordinates of this point must satisfy this equation as well as this equation

so both these equations will have to be satisfied and therefore if we try to solve these two equations simultaneously what we can do is we can replace this y by x by root three plus c

so then we get

so we basically have these two equations which must be satisfied by the coordinates x and y of the point of intersection of the straight line l with the circle c_2

so we simply put this y equal to this in this equation and we get this equation here now it is said that since l it is said that l must be a in the question it is said that l must actually be a tangent to this circle c_2

so if l is a tangent to this circle c_2 then there should be only one point of intersection between the straight line l and c_2 which means that there should be only one solution of this equation if you see here this equation is actually quadratic this equation is quadratic in x

so potentially depending on the value of c you know there could be more than there could in in general there could be two solutions of x but then the idea is that we should choose this c in such a way that there is only one solution of x

here in this equation or basically both the roots must be equal

so if we choose such a c then the corresponding line will basically touch the circle c^2 at only one place

so if we open this equation what we get is and then if we rearrange the terms we end up getting this equation here

so for the roots to be equal the condition is basically that the discriminant should be 0

so the discriminant here is going to be $2c \sqrt{3} - 6$ whole square minus $4 \times 4 \times 3$ sixteen that is sixteen by three times eight plus c^2 square

so this equation this quadratic equation has equal roots if and only if this discriminant of the quadratic equation equals zero and then we can further simplify this equation also

so we will get

so this is the first here term and then we have in this equation as we can see itself is quadratic in c

so we get

so if we take all of this on the right hand side we get $4c^2 - 8\sqrt{3}c + 20 = 0$ sorry plus eight times $\sqrt{3}$ into c plus twenty by three equals zero and then the two values of c are $c = \frac{2\sqrt{3} \pm \sqrt{3}}{2}$

so the two values of c are this n and if we simplify them further then these other two values

so these are the two values that we get for c and then let us put these values back into the equation

so the equation was of l

so the equation of the straight line l was $y = x\sqrt{3} + c$

so for the first case where c is $-\sqrt{3}$ the equation of the line l becomes $x\sqrt{3} - \sqrt{3}$ that is $x - 1$ into y equals 1

so and this matches with the possibility a over here

so a is correct let us see the other possibility where c is $-\frac{5\sqrt{3}}{3}$

so if we put it there if we put c equal to $-\frac{5\sqrt{3}}{3}$ we get the equation $x\sqrt{3} - 5$ and this unfortunately this equation is not in on any of the four options

so option b is the right choice let us take another problem now in this problem it is said that let p, q and r, s be tangents at the extremities of the diameter pr of a circle of radius r

so what we have is a circle and pr is one of its diameter radius of the circle is r and it is said that

so this is rs

so both p, q and r, s are tangents to this circle it is said that if ps and rq

so so this is the straight line ps and here we have the straight line rq

so it is said that these two points q and s are such that ps and rq intersect at a point which lies on the

so they intersect here and q and s are such that this point of intersection of ps and rq lies on the circumference of the circle

so if that happens then what can we say about the diameter pr of the circle in terms of the lengths p, q and r, s because there has to be relation

so we take this understanding on to the other to the next slide what we have is we have a circle like this let us say this is the center o we have a diameter pr the radius of the circle is r and then it is said that we have two tangents rs and p, q but it is said that these lengths

so so these two are tangents rs and p, q are tangents to the circle but then

their lengths the lengths of these two tangents are such that if i connect p to s and if i connect r to q then then those straight lines drawn in red and green will exactly intersect on a point on the circumference of the circle which is this point

so that is why i purposefully purposefully i had drawn are i had drawn this green line to pass through this point and r because that is what the question is asking that what is the

so how should we choose these two points q and s

so we are asked to find the expression of the diameter in terms of these blanks p q n r s such that these two straight lines which are drawn one is from the point p of this point p of this tangent to the point s of the other tangent and the other straight line is from the point of contact of this other tangent to the circle which is point r to the point q of the first tangent

so these two are intersecting at it is said that i will one has to choose this length in such a manner that these two lines intersect at a point and that point must be on the circumference

so that is the important condition that it must lie on the circumference of the circle

so and of course this point of intersection is called x

so here we will be using ah to solve this problem we rely we realize that the most important fact here is that these two lines intersect on a point on the circumference of the circle

so that is the most important fact that will be useful to us and because they intersect on a point on the circumference from high school we know that this angle will be equal to ninety degrees if this point of intersection of these two lines was not on the circumference of the circle then this angle might not be 90 degrees now because this is 90 let us say that this angle is theta now because this angle is theta and this is a straight line

so this is also 90 degree

so if we see this right angle triangle p x cube then we have theta and nine t

so this angle must be pi by two minus theta since p q is a tangent this angle o p q is 90 this is 90 and this part of that angle is pi by 2 minus theta means it follows that this angle must be theta and therefore now if you look at this triangle rsp which is also a right angle triangle it is right angled at r its angle rsp is going to be because this is theta this will be pi by 2 minus theta and then just by looking it is now clear that the 3 angles of this right angle triangle are ps and the three angles of rpq are the same because one is one angle is 90 degrees the other is theta and the third angle is pi by 2 minus theta in both the cases because for the triangle rpq this angle is going to be because this is 90 and this is theta this angle will obviously be pi by two minus theta and since the three angles of these two triangles are the same these two triangles are similar

so i am just drawing these two triangles separately

so i am first drawing this triangle rps

so it is right angled at r the angle at p is theta and then i am also drawing rpq which is right angle at p when in r p q this angle p q r is theta since these two triangles are similar from the similarity ratios we have that r s divided by r p must be equal to this divided by this which is r p divided by p q and from here it follows that r p square is p q times r s and therefore r p is square root of p q k times r s but r p is nothing but the diameter two r

so this basically shows that the diameter is nothing but the square root of the product of these the length of these two tangents

so and that is basically option a

so next let us see what is meant by the power of a point with respect to a

circle

so second let us consider this circle here whose center is O and suppose that we have a point P here and we define the power of this point P with respect to this circle as the squared length of the tangent from this point to this circle

so let us say that PT is the tangent PT is a tangent from this point P to the circle then the power of a point P with respect to this circle let us say C is equal to PT^2 the square of this length PT and then we will also prove a very interesting result that suppose now we we construct any straight line starting from P which cuts this circle at two points A and B

so this could be any straight line

so any arbitrary straight line

so let us say this straight line here and it this straight line cuts the circle at points A and B and then we will show that $PA \times PB$ is equal to PT^2 square and this is the power of this point P

so the product of these two lengths

so these are the lengths from the point P to the two points where the straight line has cut the circle and this is true for any straight line

so even if i had drawn let us say another line like this and if A and B were like this then $PA \times PB$ would still give me the same value of PT^2 square because remember this value PT^2 square only depends on the coordinates of this point P and then i am claiming that if i draw any straight line from P

so i just drew any arbitrary straight line then the product of these two lengths between this point P and the points where this arbitrarily drawn straight line cuts the circle

so if i take the product of those two distances then it will be equal to the power of this point P with respect to the circle

so let us prove this fact

so to prove this let us first connect these two points A and B to the center of the circle over here and let us also connect this point P to this point T then we will also connect A to A and T now let us say that this angle is θ then from high school geometry we know that the angle subtended by an arc the angle subtended by an arc let us say this arc AB at the center

so i am talking of this angle is always twice the angle subtended by the same arc at any point on the circumference

so if we take this point T then the angle subtended by this arc AB at this point T on the circumference is θ and therefore the angle subtended by the same same arc at the center is going to be 2θ similarly now consider this other arc AT the angle subtended by this arc AT at this point B on the circumference let us denote it by ϕ then again from the same result that we used earlier the angle subtended by this arc AT at the center of the circle is going to be twice this angle which is 2ϕ and now if we see this angle is obviously equal to $\theta + \phi$ because it is the exterior angle of this angle of the triangle AOB but also if we look at this triangle BOA it is an isosceles triangle because this length and this length are equal to the radius of this circle and therefore this and this angle are equal and they are equal to 90° minus θ plus ϕ

so 90° minus θ plus ϕ now since this PT is a tangent to this circle at this point T the angle PTO is 90° and therefore this angle ATP must be equal to 90° minus θ plus this other angle which is equal to ϕ

so now if we see this triangle APT this angle is $\theta + \phi$ this angle is ϕ similarly another triangle is BPT this angle is $\theta + \phi$

so the angle at B is also ϕ sorry the let us consider the triangle BTP the angle BTB is $\theta + \phi$

so we can just by looking at these two triangles we can say that triangle $t a p$ is similar to triangle $b t p$ because all the three angles are of these two triangles are equal and since they are similar from the similarity from the similarity ratios we get that $a p$ divided by $p t$ must be equal to $p t$ divided by $p v$ and then from here it is clear that $p t$ square is $p a$ times $p b$

so this proves this statement

so ah we will try to give a little example here to illustrate the fact that we had just proved in the previous slide which was $p a$ into $p b$ is equal to $p t$ square

so this is the coordinate axis

so the x and the y axis are shown we have the origin here and let us say that we have a circle whose center is here at five comma three just an arbitrary circle and let us say it has radius two

so so it is something like this and let us now consider a point p whose coordinate r whose coordinates are two minus two then clearly the power of this point p with respect to this circle will be the square length of the tangent from this point p to this circle

so so this will most likely be the tangent

so the power of this point p is the square of this length $p t$ now to find this $p t$ is not very difficult let us join p with the center of this circle at o we see that this triangle $p o t$ is a right angle triangle and therefore $p t$ square plus $o t$ square is equal to $o p$ square from the pythagoras theorem now clearly the radius of this circle is two and therefore $o t$ square is 4 because $o t$ is a radius of the circle $o p$ square is because we know the coordinates of both o and p $o p$ square is five minus two whole square plus three minus minus two whole square and this comes out to be thirty three and therefore when we use these two values into this equation we get $p t$ square equal to thirty

so the power of this point p two comma minus two with respect to this circle is thirty now let us consider some other point ah let us say nine comma five

so so this is nine comma five and let us join this $p n$ nine comma five by a straight line and lets say that this straight line obviously cuts the circle at two points a and b now we will try to find this length $p a$ and $p b$ will take their product and verify whether that product is equal to 30 or not because that is what we had shown in the previous slide now if we see this straight line over here then the equation of the straight line is that if there is any point let us say x comma y on the straight line then y minus minus 2 divided by x minus minus two must be equal to the slope of this line which is five minus minus two divided by nine minus two

so if we simplify this the equation of this line comes to y equal to x minus 4 y equal to x minus 4 now the equation

so now we essentially have to find the coordinates of these two points of intersection between this red line and the circle the equation of this circle is given by this equation to find these two points the coordinates of these two points must satisfy both this straight line equation and this equation of the circle now since y is equal to x minus 4 if we replace this y by x minus 4 we get x minus five whole square plus x minus seven whole square is four and we can see that what we have got now is the quadratic equation in x

so we will get two different values of x which will correspond to the x coordinates of these two points of intersection

so the first value of x that we obviously get is x equal to five because we put x equal to five this left hand will be equal to left hand side will compute to be four

so one solution is x equal to five and when x is equal to five the y coordinate should be one

so this is one of the points which is this point a the other solution to this equation is x equal to seven because when x equal to seven this term is zero and this is four when x is seven then y is three and this is the other point of intersection now we can easily find this distances p and p_b p_a is equal to square root of which comes out to be square root of eighteen and p_b is square root of three minus minus two whole square plus seven minus two whole square which comes out to be square root of fifty p into p_b is now square root of eighteen times fifty which is equal to thirty

so we see that indeed the product of p_a and p_b is equal to the power of this point p

so with that we come to an end to this lecture in the next lecture we will start a new topic on the common tangents to two circles thank you you