

welcome to lecture two on circles in the first lecture we have derived different forms of equations of a circle in this lecture we will continue to do that in the starting

so we will first discuss how to derive the equation of a circle which passes through three given non collinear points then we will move on to finding the equation of a circle for which the two end points of any diameter are given we will also see how to check whether a given point belongs to the interior region of a circle we will also derive formulas to find the intercepts of a circle on both the x and y axis and in between we will also solve some problems to illustrate these derivations

so let us start with the following

so suppose we are given three non collinear points

so the three points are not in a straight line then from high school we know that if we have three points and they are not in a straight line then there exists a unique circle which passes through these three points or basically a unique circle on which all these three points will lie we can always find a unique circle but only as long as these three points are non collinear

so they should not be in a straight line

so let us say that we have these three points coordinates are $x_1, y_1, x_2, y_2, x_3, y_3$ in order to find the circle through which passes through all these three points we basically just need to find out the center of that circle and also the radius of the circle

so if we can uniquely find out the center and the radius of that circle then we basically have the equation of the circle we know a simple result from high school again that if we have a circle and if we take any chord for example the chord ab then from high school we know that the center of the circle will always lie somewhere on this perpendicular bisector many times students are asked the question that suppose if there is a circle or you are given a circle and you are asked to find the center of the circle

so the center is not shown here

so how would you find the center of this circle

so one way is to use this property

so what we do is we can just construct any two arbitrary chords

so this is one chord this is another chord and then we draw we construct the perpendicular bisectors for both these chords

so this is the black dotted line is the perpendicular bisector of the chord ab and this other blue dotted line is the perpendicular bisector of the chord cd now from this property here we know that the center of the circle must lie on the perpendicular bisector of any chord therefore the center must lie on this somewhere on this perpendicular bisector of the chord ab the center of the circle must also lie on this blue dotted line which is the perpendicular bisector of this chord cd since the center must lie on both this black dotted line and the blue dotted line and these two lines intersect exactly in one place which is here we are sure that this point of intersection is the center of this circle

so this is the basic idea that we are going to use here to first find the center of the circle

so if there is a circle which passes through all these three points

so suppose there is a circle like this which passes through these three points then the straight line joining p_1 and p_2 will be a chord of this circle similarly the straight line joining p_2 and p_3 will be another chord of the same circle and then using the result that we just saw now we can easily find the center of this circle because now we have two chords we just need to

construct

so let this line be the perpendicular bisector of the chord $p_1 p_2$ and let this line be the perpendicular bisector of the chord $p_2 p_3$ and then of course the point of intersection of both these perpendicular bisectors is going to be the center of the circle once we find the center finding the radius is very easy we can just measure the distance between this centre and any three of these points

so so this

so op_1 will be equal to op_2 will be equal to op_3 all of them will be equal to the radius of this circle and once we know the coordinates of the center and the radius of the circle we can write the equation of the circle

so let us do that in a little bit of detail now to get the coordinates of the center here one way is to write down the equation of these two perpendicular bisectors and then solve for the point of intersection

so starting with the bisectors let this is bisector b_1 this is bisector b_2

so this bisector b_1 will pass through this midpoint of the chord and the coordinates of the midpoint is coordinates of the midpoint of this chord is $x_1 + x_2$ over two $y_1 + y_2$ over two similarly the coordinates of this midpoint of this other chord $p_2 p_3$ is $x_2 + x_3$ over two $y_2 + y_3$ over two

so now writing the equation of this perpendicular bisector b_1 is not very tough because suppose if we have a point here any point let us say we have a point having coordinates x and y on this perpendicular bisector b_1 then the slope of this perpendicular bisector b_1 is going to be $y - y_1$ over $x - x_1$ further we see that this chord and this perpendicular bisector b_1 intersected at 90° degrees therefore the product of the slopes of these two lines will be minus one hence slope of the perpendicular bisector times slope of this chord which is $y_2 - y_1$ over $x_2 - x_1$ is equal to minus one

so if we rewrite this nicely we will basically get the equation of this perpendicular bisector because the x and y coordinates of any point on this perpendicular bisector has to satisfy this equation and if we refine this a little bit we get this times

so this is what we get after we refine this

so this is the equation of the first the equation of the perpendicular bisector b_1 of chord $p_1 p_2$ in a similar manner the equation of the perpendicular bisector of $p_2 p_3$ is going to be and that we can write straight away because it is going to be a similar expression $y - y_2$ over $x - x_2$ equals zero and now we just need to solve these two equations simultaneously because we know that the center of the circle is the point of intersection of these two perpendicular bisectors whose equations are given by these two equations

so to find the center of this circle we just need to these two equate linear equations simultaneously and let the x and y which is the solution of these two equations let us represent that by x_0 and y_0

so y_0 and x_0

so then x_0 and y_0 is the center of the circle which passes through $p_1 p_2$ and p_3 now once we have found the coordinates x_0 and y_0 of the centre of this circle

so the radius will be given by $r = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$ and as I said earlier the radius will be the same whether we take this distance or this distance or this distance then of course once we have got the radius and the center we can easily write the equation of this circle in the center radius form the other important thing was

that why do we really need this condition of non collinearity of these three points and it should not be very difficult to see that because let us say that what if these three points were collinear

so if these three points were only collinear that is that they were on the same straight line then we will show that they can never lie on the same circle if they were they were to lie on some circle then the center of that circle must be at the intersection of the perpendicular bisector of both this chord and the perpendicular bisector of this chord and that we have already seen earlier but if we see here the perpendicular bisector b_1 of the chord $p_1 p_2$ is like this and the perpendicular bisector of the chord $p_2 p_3$ which is b_2 is this other blue dotted line but because $p_1 p_2 p_3$ lie on the same straight line it is very easy to see that both the perpendicular bisectors are parallel to each other because we have a straight line here and this is and this is also ninety and hence these two perpendicular bisectors are parallel and therefore they will never intersect and since they will never intersect it means that there cannot be any circle because then there we cannot find a point where these two intersect because they are parallel and hence there will not be any circle through which all these three would pass because if there is a circle from where these three will pass then the perpendicular bisectors must intersect at the center of that circle but since they are not intersecting these two perpendicular bisectors are not intersecting in this case where the points are collinear it follows that there will exist no circle on which all these three points will lie we will take up a short example to illustrate this procedure

so in this example we are given four points and we are asked to show that they are concyclic concyclic means that all these four points they lie on the same circle

so one way of showing that is that we take let us say the first three points and we check if they are collinear or not if they are non collinear then we can find the equation of a circle which passes through these three points once we have the equation of that circle we can simply check whether this point the fourth point here lies on that circle or not

so the step one the first step is to first find the equation of a circle through which which passes through these three points

so so one of the points is we if we were to draw the coordinate axis with the origin here

so the point one zero is over here let us say

so let us say each square here is a is of unit length point two minus seven will be somewhere on this vertical line

so it is somewhere here because this is seven units and the third point is eight one which is

so these are the three points

so as discussed if a circle has to pass through these three points let us name them $p_1 p_2 p_3$ then we can consider these two straight lines which will obviously be the which will be chords of the circle which passes through $p_1 p_2$ and $p_2 p_3$ and we will find out the equation of the perpendicular bisectors of these two chords let's say bisector b_1 and bisector b_2 and then we will see where they intersect

so this point of intersection will be the center $x_0 y_0$ of the circle which passes through $p_1 p_2$ and p_3 and of course in this case we geometrically we can see that these three points are not collinear and that is why we proceeded further with finding the circle through which which on which all of them lie for the first perpendicular bisector any point having coordinates $x y$ on b_1 will have a slope equal to

so so any point $x y$ on b_1 must satisfy the following equation

so the midpoint of this chord here is one point five and minus three point five and therefore the slope of this perpendicular bisector is $y - (-3.5)$.

5 which is $y + 3.5$.

5 over $x - 1.5$.

5 and since the this bisector perpendicular bisector and the chord are at 90° degrees the product of the slope of the perpendicular bisector which is this expression times the slope of the chord $\frac{y - 1}{x - 2}$ which is will be $-\frac{7}{2}$ divided by $\frac{y + 3.5}{x - 1.5}$ which is $-\frac{7}{2}$ the product must be -1 and therefore the equation of this is the equation of this perpendicular bisector of this chord $\frac{y + 3.5}{x - 1.5}$ is

so this is the equation of the line b_1 which is the perpendicular bisector of $p_1 p_2$ in a similar manner we will find the equation of the perpendicular bisector of this other chord $p_1 p_3$ the coordinate of this midpoint of this chord will be $(4, 0)$.

5 comma 0 .

5 now we have any point having coordinates x and y on this bisector of the chord on this perpendicular bisector of the chord $p_1 p_3$ then the slope of this perpendicular bisector b_2 is $\frac{y - 0}{x - 4}$.

5 since this perpendicular bisector and the chord are at 90° degrees the product of their slopes will be -1 therefore this times the slope of the chord which is $\frac{y - 1}{x - 2} = -1$ we have $y - 1 = -x + 2$ equals $y = -x + 3$

so this is the equation of the bisector b_2 which is this dotted green line and now this center is the center is at the point of intersection of these two perpendicular bisectors there is a little correction here this will be $(3, 1)$.

5

so from the previous slide we get that the coordinates of the center of this circle satisfy these two equations and then from these two what we see is that $x = 3$

so i take this $(3, 1)$ on this side i just have $y = 1$ here in this equation

so $y = 1$ is basically equal to this similarly from the second equation if i take $x = 3$ to this side i have $y = -3 + 3 = 0$ which is $y = 0$

so both these are $y = 1$ and therefore they have to be equal and then we can get that $5 \times 1 = 7$ equals

so we have to take this on this side

so we get from where we get that the x coordinate of the center of the circle is 3 and therefore the from this equation or basically from this equation we have $y = -x + 3$ equals $-3 + 3 = 0$ just minus three the center of the circle is at $(3, 1)$ and now that we know the center of the circle

so this is $(3, 1)$ you can easily find the radius it will be this distance

so radius r is that comes out to be 5 units and then it is very easy to write the equation of the circle

so it will be $(x - 3)^2 + (y - 1)^2 = 25$ since $(3, 1)$ we get this as the equation of the circle $r = 5$

so r^2 is 25

so this is the equation of the circle on which these three points lie and now we just need to check whether this fourth point lies on this circle

so if we put $x = 9$ $y = -6$ on the left hand side we get $16 + 49 = 65$ which is 25

so the left hand side is equal to the

so the left hand side comp we computed that the with x y equal to 9 and minus 6 we see that the left hand side is 25 which is equal to the right hand side in the equation of the circle and hence this point indeed lies on the circle on this circle which passes through the first three points and hence all the four points lie on that circle

so with regards to the previous problem where we were given three points which were non coniliar and we were asked to find the equation of a circle which passes through all these three points

so apart from the method that we discussed another method is as follows

so in this other method we use the general form of the equation of a circle which is given by this equation and we have already seen this in the previous lecture since these three points lie on this circle this equation must be satisfied by the coordinates all these three points since this equation must be satisfied by the coordinates of the first point p_1 if we replace x and y by x_1 and y_1 we should still get a zero on the right hand side since p_1

so this is the circle capital c since p_1 lies on c it follows that $x_1^2 + y_1^2 + 2g x_1 + 2f y_1 + c = 0$ and similarly since p_2 and p_3 also lie on this circle we get other two equations

so for p_2 we get this equation and for p_3 we get this third equation obviously to completely describe the circle we have three unknowns and they are g f and c and if you see here what we have is three equations all of them linear in g f and c hence here we have a linear system of equations in three unknowns three equations and therefore we should be able to solve it

so on solving we will get the values of g f and c and when we put these values back in this equation we get the complete equation of the circle in the general form we consider another way of describing the equation of a circle

so suppose that we have a circle where we are only given the two end points x_1 y_1 and x_2 y_2 of some diameter of that circle and then we are asked to find the equation of this circle

so one way to do that is if we again go to our high school geometry then we know that if we have any point x y on the circle on the circumference of the circle and if we connect this point to the two end points of this diameter then we know that this angle here is always 90 degrees

so we will use this property to derive the equation of the circle since the angle here is 90 degree the product of the slope of this cord times the slope of this curve should be minus one the slope of this chord is $\frac{y_2 - y_1}{x_2 - x_1}$ times the slope of this chord is $\frac{y - y_1}{x - x_1}$ then this product should be minus one and then if we rewrite it ah nicely what we get is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

so as we can see this if we expand this further then we will see that we get this equation $x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0$ and this is exactly in the general form of a circle which was $x^2 + y^2 + 2g x + 2f y + c = 0$ and by looking at it we can also find out the coordinates of the centre and the radius of the circle

so the center is $(-\frac{g}{1}, -\frac{f}{1})$ will be $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

so the center is $(-\frac{g}{1}, -\frac{f}{1})$ and g by comparing these two g is equal to $-\frac{x_1 + x_2}{2}$ f is $-\frac{y_1 + y_2}{2}$ and similarly we can find the radius also lastly let us take this problem of how to check the position of a point of an arbitrary point with respect to a circle

so suppose we are given some point those coordinates are a and b
so we have a point p with coordinate a and b and we have a circle whose equation in the general form is $x^2 + y^2 + 2gx + 2fy + c = 0$

so now how can we tell whether this point p lies inside the circle or it lies outside the circle

so geometrically if we see this circle then we see that circle has center at $(-g, -f)$ and radius equal to square root of $g^2 + f^2 - c$ now if this point a, b is somewhere there on this plane

so if this point a, b is inside the circle let us say here then it is clear that the distance between this point p and the center will be less than the radius r

so p is inside the circle if and only if the distance between the point p and the center of the circle which is this is less than the radius

so the radius is $r = \sqrt{g^2 + f^2 - c}$ and if we simplify this further then this condition is essentially equal basically $a^2 + b^2 + 2ga + 2fb + c < r^2$ but if you see this left hand side is nothing but this second degree equation here with x equal to a y equal to b

so to check if the point is inside when this point is inside we just replace x and y by the coordinate a and b respectively and then we check if the value that we get if it is less than zero if this value after replacing x and y with a, b if the value that we compute is less than zero then it means that the point is inside the circle similarly if the point is outside the circle then the distance will be greater than r and then the condition that we will get is that again this value here should be greater than zero and then of course if this point p is lies on the circle then this will be exactly equal to zero

so these are the three scenarios given a point a, b and a circle c having this equation we will first compute this expression with the coordinates of the point p

so we get $a^2 + b^2 + 2ga + 2fb + c$ equals

so we if we if we replace x with a and y with b in this left hand side we get this value here and then we check this value because this value will be either positive or negative or equal to zero now if this value equals zero

so if equals zero then it follows that p lies on the circle c if this value is less than zero then it follows that p lies inside c and finally if this value is greater than 0 then it follows that point p lies outside the circle c

so if this is true then p lies outside the circle c

so with that we will finish this lecture in the next lecture we will continue with all the other topics like finding the intercepts made by a circle on the axis on both the axis will also derive conditions to check whether a line passes through a circle or it does not or it passes through the center of the circle or not and will also take up some problems thank you