

welcome students to the fifth and final lecture on infinite series in the last lecture we have developed the concept of Euler's number as you know it is denoted by e and we have discussed that it can be obtained as the limit n going to infinity $1 + \frac{1}{n}$ to the power n . Let me do this once again to show how it converges to an infinite series which summation will lead to the Euler constant e .

So consider $1 + \frac{1}{n}$ to the power n . The k th term for k less than n can be obtained from binomial theorem. The k th term for k is equal to $\binom{n}{k} \frac{1}{n^k}$. This is equal to $\frac{n!}{k!(n-k)!} \frac{1}{n^k}$. After cancelling with this we get $\frac{n(n-1)\dots(n-k+1)}{k!} \frac{1}{n^k}$. Now let us divide each one of the k terms by $1/n$.

So what we are getting this is equal to $1 + \frac{1}{n} + \frac{1}{2!} \frac{1}{n} + \frac{1}{3!} \frac{1}{n} + \dots$. As n goes to infinity this whole expression converges to $1 + \frac{1}{k!}$ as each of $\frac{1}{n^2}$ and $\frac{1}{n^{k-1}}$ for a fixed k go to 0. Therefore what we are left with is $1 + \frac{1}{k!}$. Therefore the limit $1 + \frac{1}{n}$ to the power n this converges to first of all notice that if n increases and n goes to infinity then the number of terms that you will find in the series is also going to infinity and the k th term is going to be $\frac{1}{k!}$. Therefore the 0th term is 1 , the 1st term is $\frac{1}{1!}$, the 2nd term is $\frac{1}{2!}$, the 3rd term is $\frac{1}{3!}$. Therefore as n increases the k th term converges to $\frac{1}{k!}$. Therefore the infinite series can be seen as it is the sum of $\frac{1}{k!}$.

So that is $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$ that is one plus one upon one factorial plus one upon two factorial plus one upon three factorial plus this is called Euler's number. Question is what is the value of it? This is an irrational number therefore we cannot write e as the ratio of two integers but we can find the limit of it which is two point seven one eight two eight one eight two eight four five nine zero four five two three zero three six five.

So I remember up to this position for all practical purposes we typically use up to three or four decimal places and people have tried to compute its value through computers but since computers also have a limit of the number of decimal places for all practical purposes we have to take an approximation to that one and therefore using a third, third or fourth decimal places after the decimal point is good enough for our practical purpose. Now it is very easy to find out that this number has to be greater than 2 because of the first two terms this is 2 and the remaining sum it is an infinite sum but all the terms of this summation are positive real numbers in fact positive rational numbers it is $\frac{1}{k!}$ therefore it is easy to see that this part is going to be positive it cannot be negative therefore e has to be greater than two that is very clear. How do we know that it is not greater than three?

So we have to see that the bound on these infinite sum is one. Let us verify that.

So consider $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is equal to $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$ like that now replace all other numbers with 2.

So this is less than $1 + \frac{1}{2}$ this quantity is less than $1 + \frac{1}{2}$ because 3 is greater than 2 therefore $\frac{1}{3}$ is smaller than $\frac{1}{2}$.

plus $\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$ like that now this is a geometric series of the form $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ plus half cube plus half to the power four like that and we can sum this geometric series as n goes to infinity this

converges to half into one upon one minus half which is equal to one therefore we have seen that is greater than 2 but the part that is remaining above 2 is less than 1 therefore we can easily say that 2 is less than e less than two plus one is equal to three therefore we know that it is between two and three and the actual value was that i have shown some time back 2.

7 1828 1828 etcetera now let us consider e to the power x for some real or complex x question is what is going to be the series i am not going to prove but i am just writing the result that e to the power x is equal to 1 plus x plus x square upon factorial 2 plus x cube upon factorial 3 plus this infinite sum is called e to the power x let me give you an intuitive idea how it can consider e square we know that e square is equal to e multiplied by e therefore we can write it as limit n goes to infinity 1 plus 1 upon n whole to the power 2 n is equal to limit n goes to infinity 1 plus 1 by n whole square whole to the power n is equal to limit n goes to infinity one plus two by n plus one by n square whole to the power n

so the kth term for zero less than equal to k less than equal to n is n c k two by n plus 1 by n square whole to the power k is equal to factorial n factorial k factorial n minus k 1 upon n whole to the power k into 2 plus 1 upon n whole to the power k is equal to by using the same trick that we did sometime back with respect to e this is becoming 1 into 1 minus 1 upon n up to 1 minus k minus 1 upon n upon factorial k 2 plus 1 upon n whole to the power k therefore when we take limit for a fixed k this limit becomes one upon factorial k two to the power k therefore e square is actually the summation whose kth term is 2 to the power k upon factorial k therefore it is 0th term 2 to the power 0 upon factorial 0 plus 2 to the power 1 upon factorial 1 plus 2 to the power k upon factorial k or we get the series like this one plus two upon factorial one plus two square upon factorial two two to the power k upon factorial k this is not a proof but this shows how e square can be written as an infinite series involving one plus two upon factorial one plus two square upon factorial two etcetera up to infinity

so this gives an idea that e to the power x is one plus x plus x square upon factorial two x cube upon factorial three etcetera up to infinity therefore what is e to the power minus x by replacing x with minus x we can easily get that it is 1 minus x plus x square upon factorial 2 minus x cube upon factorial 3 plus this infinite sum where alternate terms will come out to be positive and negative let us now consider a complex number purely imaginary e to the power i x all of you know about i i is the root over minus 1 and we use it for denoting complex numbers in fact a plus i b all of you are very familiar with by expanding in the similar way we get this is 1 plus i x plus i x square upon factorial 2 plus i x cube upon factorial 3 plus i x to the power 4 upon factorial 4 plus i x to the power 5 upon factorial plus i x to the power six upon factorial six etcetera we know i square is equal to minus one therefore this can be written as 1 plus i x i square is equal to minus 1

so minus x square upon factorial 2

so i cube is equal to minus i x cube upon factorial three i to the power four is equal to one therefore it is x four upon factorial four plus i x to the power five upon factorial five etcetera let us now separate out the real terms and imaginary terms

so what we get is one minus x square by factorial two plus x to the power four upon factorial four minus x to the power 6 upon factorial 6 etcetera plus i times x minus x cube upon factorial 3 plus x to the power 5 upon factorial 5 etcetera now do you recognize these two series separately do you about two classes back we have discussed that this is nothing but cos x and this is this is sine x therefore what we can see that e to the power i x can actually be

written as $\cos x + i \sin x$ ok let us move further let me solve an example find the value of $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ what it is going to be we can see that it is not exactly what is written for e because for e it is $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ but what we can do we can write it as $\frac{1}{1!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ we can cancel the two plus three cancels with three is equal to one upon two factorial plus one upon three factorial plus like that therefore we can see that this is also equal to e what about the next problem if we have $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ up to infinity we can easily see that it is x taken out it is $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is equal to x times e let me do a slightly different problem what is $\sum_{n=0}^{\infty} \frac{1}{n!}$ we know that at $n=0$ is equal to zero the denominator is equal to minus one factorial and minus one factorial does not have any meaning therefore we can write it as $\sum_{n=1}^{\infty} \frac{1}{n!}$ and we can easily see now that this is equal to $\sum_{m=0}^{\infty} \frac{1}{m!}$ where m is equal to $n-1$ therefore if we are summing $\frac{1}{n!}$ from $n=1$ to infinity we get that is also equal to e if we move further and suppose we want to find out what is $\sum_{n=0}^{\infty} \frac{1}{(n-2)!}$ if that is the problem then as before this is actually equal to $\sum_{n=2}^{\infty} \frac{1}{(n-2)!}$ which is at $n=2$ it is one upon zero factorial at $n=3$ it is equal to three this gives us one upon one factorial and at $n=4$ it is one upon two factorial like that therefore that is also equal to e therefore we can find an expression that that is apparently different from the standard expansion of e but we can make some algebraic manipulation to convert it into e or some of its function for example what is $\sum_{i=0}^{\infty} \frac{i^2}{i!}$ this is equal to zero it is zero this is we can write i^2 is equal to one to infinity i^2 upon $i!$

so what is the k th term k^2 upon factorial k which is k upon $k-1$ factorial which is equal to $k-1 + 1$ upon $k-1$ factorial which is equal to $k-1$ upon $k-1$ factorial plus 1 upon $k-1$ factorial which is equal to 1 upon $k-2$ factorial plus 1 upon $k-1$ factorial now just now we have seen that this when we sum for k is equal to 2 to infinity is going to be e and this when we sum from 1 to infinity this is going to be e therefore the whole sum is going to be $e + e$ is equal to twice e therefore we can see that summation of i^2 upon $i!$ is equal to twice e slightly more harder problem find the value of $\sum_{n=0}^{\infty} \frac{n^3}{n!}$ n is equal to one to n is equal to zero to infinity

so this we can write it as summation n is equal to one to infinity n^2 upon $n-1$ factorial which is equal to $\sum_{n=1}^{\infty} n^2$ upon $n-1$ factorial which is $n^2 - 2n + 1$

so we need to compensate for that

so it plus two $n-1$ divided by $n-1$ factorial this is equal to $\sum_{n=1}^{\infty} (n^2 - 2n + 1)$ upon $n-1$ factorial

so one $n-1$ cancels

so it is $n-1$ upon $n-2$ factorial plus 2 times summation n upon $n-1$ factorial minus summation 1 upon $n-1$ factorial n is equal to 1 to infinity here again we make the manipulation

so it is $\sum_{n=2}^{\infty} \frac{n-1}{n-2!} + 2 \sum_{n=1}^{\infty} \frac{n}{n-1!} - \sum_{n=1}^{\infty} \frac{1}{n-1!}$

factorial

so this is going to be $\sum_{n=3}^{\infty} \frac{1}{n!} + \sum_{n=2}^{\infty} \frac{1}{n!} + 2 \sum_{n=1}^{\infty} \frac{1}{n!}$ cancels with $\sum_{n=1}^{\infty} \frac{1}{n!}$

so $\sum_{n=2}^{\infty} \frac{1}{n!} + 2 \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=1}^{\infty} \frac{1}{n!}$ sometime back we have seen that $\sum_{n=1}^{\infty} \frac{1}{n!}$ that gives rise to e $\sum_{n=2}^{\infty} \frac{1}{n!}$ that also gives rise to e therefore similar way we can find that $\sum_{n=3}^{\infty} \frac{1}{n!} + \sum_{n=2}^{\infty} \frac{1}{n!}$ will give an e

so we find $e + e$ this gives us $2e$ this gives us $2e - e$

so what we are left with is $e + e + 2e - e - e + e + 2e - 2e - e$

so what is that this is equal to $5e$

so the summation $\sum_{n=1}^{\infty} \frac{n^3}{n!}$ is equal to $5e$ one more problem consider $\sum_{n=1}^{\infty} \frac{1}{n!} + \sum_{n=2}^{\infty} \frac{2}{n!} + \sum_{n=3}^{\infty} \frac{3}{n!}$ what is the value of this series we can see that the k th term is $\sum_{i=1}^k \frac{1}{k!}$ divided by $k!$ which is $\frac{k+1}{k!} = \frac{1}{(k-1)!} + \frac{1}{k!}$ which is equal to half times $\frac{1}{(k-1)!} + \frac{1}{k!}$ therefore if we take sum can be written as half $\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{n!}$ and we have already seen that this converges to e this converges to e and this converges to e therefore the whole series converges to $3e$ let us now look at a slightly different problem suppose you are asked to find the coefficient of x^4 in $(1+x+x^2)^e$

so we proceed in the following way we take the series expansion of $(1+x+x^2)^e$ and that we multiply by the second degree polynomial $(1+x+x^2)^e$ is $\sum_{n=0}^{\infty} \frac{(1+x+x^2)^n}{n!}$ etcetera now we try to find in how many different ways x^4 can be formed

so one multiplied by x^4 this will give x^4 and the corresponding coefficient is $\frac{1}{4!}$ this x multiplied by x^3 will give rise to x^4 and therefore the corresponding coefficient is going to be $\frac{2}{3!} + \frac{3}{2!}$ plus $\frac{3}{2!}$ x^2 into x^2 will give us $\frac{3}{2!}$ plus $\frac{3}{2!}$ ok this is equal to

so it is $1 + 8 + 36$ which is equal to 45

let me do a slightly different problem where \ln is natural log $\ln 3$ is equal to \log to the base e

so what is going to be the value of the infinite series which is of this form we can easily see that it is the $e^{\ln 3}$ right because the expansion pattern is like e^x

so this is equal to $e^{\ln 3}$ and we know that this is equal to 3 and $e^{\log 3}$ is equal to 3^e therefore this infinite series adds to 3^e

so let me do the final problem on this topic find the expansion for $e^x \cos x$ we already know the expansion of e^x we already know that expansion for $\cos x$ but what is going to be the expansion for $e^x \cos x$ for such problems we have to go in the following way let the corresponding series be $c_0 + c_1 x + c_2 x^2 + c_3 x^3$ this in finite polynomial and we need to find out the individual

coefficients c_0, c_1, c_2 up to infinity therefore we can see that e^x can be written as the product of $\cos x$ times this polynomial

so we have $e^x = \cos x \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \right)$ now $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ multiplied by $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

so we can find out the coefficients of individual powers of x from the product of the two polynomials and then equating it with the corresponding coefficient in the expansion of e^x we can obtain the values of c_0, c_1, c_2 etcetera

so let me do for first few power suffix when it is x^0 we find the coefficient on this side is one on this side it is c_0 into one therefore it implies $c_0 = 1$ now let us consider x^1 its coefficient on this side is one on this side the coefficient of x to the power one is c_1 multiplied by one implies that $c_1 = 1$ what is the coefficient of x^2 on this side we have one upon two on this side we can get x^2 as c_2 times one minus c_0 by two what does it imply it implies that $c_2 = \frac{1}{2}$ therefore $c_2 = \frac{1}{2}$ let me go one more step for x^3 on this side we have one upon factorial three on this side we have c_3 minus c_1 by two implies one by six is equal to $c_3 - \frac{1}{2}$ therefore $c_3 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ you are getting that the coefficients $e^x = \cos x \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \right)$ $c_0 = 1, c_1 = 1, c_2 = \frac{1}{2}, c_3 = \frac{2}{3}$ in fact you can try to find out c_4, c_5 is equal to $\frac{1}{24}, c_5 = \frac{1}{120}$ etcetera

so by comparing the coefficients of two series when one is known we can obtain the coefficients of the other series for which the coefficients are unknown ok students with that i conclude my lectures on exponential series hope i have taken care of varieties of problem and that will help you solving problems on series expansion thank you