welcome students to the fourth lecture on in finite series in the previous class we have seen the taylor series expansion for a function f x as f a plus f prime a into x minus a upon factorial 1 plus f double prime a into x minus a whole square upon factorial 2 like that the kth term being x minus a whole to the power k upon factorial k

so this infinite series which is a polynomial in x is an approximation of the function value at x when the function is infinitely differentiable at the point a if we take instead of the infinite series up to some nth power then we get a polynomial approximation of f x but there will be an error term and that error term goes to 0 as k goes to infinity that is the higher will be the degree of polynomial the closer will be the approximation of f x in the last class we have also seen the taylor series expansion for sin x cos x tan x this trigonometric functions we have approximated using their corresponding taylor series by expanding them up to some finitely many terms in today's class let us start with an inverse trigonometric function that is tan inverse x

so find the taylor series approximation of tan inverse x up to x to the power five we can do it even beyond that but this will give us the idea how to work out this problem

so f x is equal to tan inverse x therefore if zero is equal to zero it is clear that we have taken a to be zero that is we are expanding the series about zero f prime x is equal to one upon 1 plus x square is equal to 1 plus x square to the power minus 1 therefore f prime at zero is equal to one let us calculate the second derivative let me write it f two x which is d d x of one plus x square to the power minus one is equal to minus 1 plus x square to the power minus 2 into 2 x is equal to minus two x into one plus x square to the power minus two therefore therefore if double prime x is equal to zero at x is equal to zero the third derivative of f is equal to d d x of minus two x into 1 plus x square whole to the power minus 2 is equal to minus 2 into 1 plus x square whole to the power minus 2 plus minus 2 x into minus 2 into 1 plus x square whole to the power minus 3 into 2 x is equal to minus 2 into one plus x square whole to the power minus two plus minus two into minus two is equal to plus four x into two x is equal to eight x square into one plus x square whole to the power minus three therefore the third derivative at zero is equal to we see that this term becomes zero when we put the value of x is equal to zero but here by putting x is equal to 0 we get minus 2 therefore the fourth derivative at x is equal to d d x of minus two into one plus x square whole to the power minus two plus eight x square into one plus x square whole to the power minus three is equal to four into one plus x square whole to the power minus 3 multiplied by 2 x that is from the first term from the second term we get actually it is a product of two terms

so we will get sixteen x into one plus x square whole to the power minus 3 plus 8 x square into 1 plus x square whole to the power minus 4 into two x which is is equal to eight x into one plus x square whole to the power minus three plus sixteen x into 1 plus x square whole to the power minus 3 plus 8 x cube into 1 plus x square whole to the power minus 4 this is is equal to 24 x into 1 plus x square whole to the power minus 3 plus 8 x cube into one plus x square whole to the power minus four therefore f 4 at 0 is equal to putting the value of x is equal to 0 here we get this is 0 here also it is 0 therefore the whole term becomes 0 in order to get the approximation of polynomial of degree five we have to differentiate it once more therefore f five x is equal to d d x of twenty four x into one plus x square whole to the power minus three plus eight x cube into one plus x square whole to the power minus 4 and this is going to be 24 into 1 plus x square whole to the power minus 3 plus 24 x into 1 plus x square whole to the power minus 4 into 2 x plus other terms and you can see that since it is x cube its derivative will also have an x and since it is 1 plus x square

its derivative will also have an x therefore the derivative of the product will always contain an x in both the terms and therefore by putting 0 this will definitely give us 0.

 so therefore the only non zero term that this can produce is equal to twenty four into one plus x square to the power minus three this will also give us zero therefore f five at 0 is equal to 24 since we said that we will expand it up to a 5th degree polynomial
 so if we recollect we can see that f 0 was 0 f prime 0 was 1 f 2 at 0 was 0 f 3 at 0 was minus two f force at zero was zero and f five at zero was twenty four therefore the fifth degree polynomial approximation for tan inverse x is now you can easily understand it is going to be x minus 2 x cube upon factorial three plus twenty four x to the power five upon factorial five which on simplification becomes x minus x cube by 3 plus x 5 by 5
 so that is the approximation of tan inverse x when we go up to the 5th degree polynomial now you can do it in a slightly more tricky way if you notice that d d x of tan inverse x is equal to one upon one plus x square the expansion of one upon one plus x square which is nothing but one plus x square whole to the power minus 1 and this is going to be 1 minus x square plus x to the power 4 minus x to the power 6 like that this we know from the expansion of 1 plus x whole to the power minus one and that is going to be one minus x plus x plus x square minus x cube replacing x by x square we get this series now let us integrate both sides therefore 1 upon 1 plus x square d x is equal to by integrating this term by term we get integration of d x minus integration of x square d x plus integration of x four d x etcetera plus a constant c now the left hand side will give us tan inverse x and right hand side will give us x minus x cube by 3 plus x 5 by 5 plus c where c is the constant by putting x is equal to zero we find that c is equal to 0 therefore the desired expansion of tan inverse x is x minus x cube by three plus x to the power five upon five once we have done with tan inverse x it gives us the impetus to do similar thing with respect to other trigonometric functions for instance consider sine inverse x what is going to be its taylor series expansion we know that d d x of sin inverse x is 1 upon root over 1 minus x square this is is equal to 1 minus x square whole to the power minus half and we know the expansion of one minus x whole to the power minus half and from there we should be able to get the time series expansion of sine inverse x we proceed in the following way let us first expand one minus x square whole to the power minus half this is is equal to 1 plus minus half into minus x square plus minus half into minus half minus 1 upon factorial 2 into minus x square whole square plus minus half minus half minus 1 into minus of minus 2 upon factorial 3 into minus x square whole cube this is is equal to 1 plus x square by 2 plus 1 into 3 upon 8 x to the power 4 plus 1 into 3 into 5 upon 8 into factorial 3 x to the power 6 etcetera which is is equal to 1 plus x square by 2 plus 3 by 8 x to the power 4 plus 15 upon 48 x to the power 6 therefore now we integrate both the sides therefore integration of root over one minus x square dx by integrating term by term we get x square by two d x plus three by eight x to the power four d x plus fifteen upon forty eight x to the power six d x etcetera plus a constant c or sine inverse x is equal to x plus x cube upon factorial 3 plus 3 x to the power 5 upon 40 plus fifteen upon forty eight by seven x to the power seven plus c putting x is equal to zero we get c is equal to zero therefore the taylor series expansion for sine inverse x is x plus x cube by factorial three plus three x to the power five upon forty plus 5 to 16 into 7 x to the power 7 this we get when i am expanding it up to the seventh power of x the obvious question is therefore what is going to be the expansion for cos inverse x

so we can find it from the first principle by computing f prime f double prime f triple prime etcetera we can find it out from the expansion of sine inverse x because we know that sine inverse x is equal to pi by 2 minus cos inverse x or cos inverse x is equal to pi by 2 minus sin inverse x therefore by inserting the value of sin inverse x we can get that cos inverse x is equal to pi by 2 minus x minus x cube upon factorial 3 minus 3 x to the power 5 40 minus 5 into sixteen into seven x to the power seven like that

so from one result we can easily derive some other results if we know their mutual relationship let us now look at some other function for instance let us consider log of 1 plus x

so let us start with this f x is equal to log of 1 plus x therefore f 0 is equal to log of 1 is equal to 0.

f prime x is equal to 1 upon 1 plus x therefore f prime 0 is equal to 1 f double prime x is equal to derivative of this which is is equal to minus one plus six whole to the power minus 2 therefore f double prime at 0 is equal to minus 1 f triple prime x is equal to derivative of minus one plus x whole to the power minus two which is going to be two into one plus x to the power minus three therefore f triple prime at zero is equal to two the fourth derivative of x is going to be minus 6 into 1 plus x whole to the power minus 4 therefore f 4 at 0 is equal to minus 6 therefore we can see that log of 1 plus x can be expanded as x minus x square by 2 plus two x cube upon factorial three minus six x to the power four upon factorial 4 is equal to x minus x square by 2 plus x cube by 3 minus x to the power 4 upon 4 like that

so it is alternatively minus and plus and this summation is therefore the series is sigma x to the power k upon k into minus 1 to the power k minus 1 which will ensure that every alternate term is going to minus and plus k is equal to one to infinity

so this is the expansion of log of one plus x but if we since we know expansion of one upon one plus x we can attempt the problem in a different way one upon one plus x is equal to one plus x whole to the power minus one is equal to one minus x plus x square minus x cube plus x to the power 4 like that therefore by integrating both sides minus x d x plus x square d x minus x cube d x plus x to the power 4 dx plus c therefore log of 1 plus x is equal to x minus x square by 2 plus x cube by 3 minus x to the power 4 by 4 plus 6 to the power 5 by 5 etcetera

so we can see that we can get the same answer even in this way also only thing left is to determine the value of c at x is equal to zero log of one plus x is equal to log of one is equal to zero therefore c is equal to zero therefore we get the above series as the expansion of log of one plus x now by putting x is equal to minus x we get log of 1 minus x is equal to minus x minus x square by 2 minus x cube by 3 minus 6 to the power 4 by 4 etcetera that is the all the terms here comes out to be negative sign once we have the expansion for log one plus x let us consider the next problem what is the expansion of log of cos x log of cos x for x belonging to minus pi by 2 to pi by 2 we know that in this range cos x is going to be positive therefore ah log is valid let me do it from the first principle f of x is equal to log of cos x therefore f at zero is equal to log one is equal to zero if prime x is equal to one upon cos x into minus sin x is equal to minus tan x therefore f prime at zero is equal to zero f double prime x is d d x of minus tan x is equal to minus one plus tan square x therefore f 2 at 0 is equal to minus 1 the third derivative is equal to minus two tan x into one plus tan square x is equal to minus two tan x minus two tan cube x therefore if third derivative at 0 is equal to 0 because if we put x is equal to 0 then this becomes 0 and this becomes 0 as well therefore the fourth derivative of x is equal to minus 2 into 1 plus tan square x minus 6 tan square x into 1 plus tan

square x is equal to minus 2 minus 2 tan square x minus 6 tan square x minus 6 ten to the power four x is equal to minus two minus eight tan square x minus six tan four x therefore f four at zero is equal to minus two in a similar way f five at x is equal to minus sixteen i am differentiating this with respect to x
 so minus sixteen tan x into one plus tan square x minus twenty four tan cube x into one plus tan square x is equal to minus sixteen tan x minus 16 tan cube x minus 24 tan cube x minus 24 10 to the power 5 x or f five x is equal to minus sixteen tan x minus forty tan cube x minus 24 then 5 x therefore f 5 at 0 is equal to 0 as all these becomes zero at x is equal to zero let me go one more step
 so what is going to be f six of x this is going to be minus 16 into 1 plus tan square x plus other terms which we by now we know that all of them will become 0 when we put x is equal to zero therefore f six at zero is going to be minus 16 therefore we find that for log of cos x f at zero is equal to zero f prime at zero is equal to zero if second derivative as zero is equal to minus one third derivative at zero is equal to zero the fourth derivative at zero is equal to minus two f five at zero is equal to zero and sixth derivative at zero is equal to minus sixteen therefore the series expansion is minus x square upon 2 minus 2 into x to the power 4 upon factorial 4 minus 16 into x to the power 6 upon factorial 6 etcetera which is nothing but minus x square by 2 minus factorial 4 is equal to 24
 so it is x to the power 4 upon 12 minus x to the power six upon it is sixteen sixteen upon factorial six which is seven twenty
 so x to the power 6 upon 45 etcetera
 so that is the series expansion for log x log of cos x you can also do it in a different way we can write log of cos x is equal to log of 1 plus cos x minus 1 this term in minus pi by two to pi by two will always remain between zero and one and therefore this expansion is valid and we have already found the expansion of log of 1 plus x for mod x less than 1 and therefore i this can be expanded by considering this as the term and expanding it using the expansion of log i leave it as an exercise for you to practice now let us come to one of the most important concepts of mathematics that is euler's constant e all of you have been using e because the natural logarithm that we take is against e and e is defined as limit n goes to infinity 1 plus 1 by n whole to the power n you may wonder where from this term has come
 so let me give you a brief idea suppose there is a time period t in which the money that you deposit here at this time point gets doubled at this time point therefore if you put x amount of money here at this point the money becomes x into 1 plus 1 that is it gets doubled
 so this is the interest rate over this time period now this money will not remain the same if we pay the interest at an intermediate time suppose we decide to give some interest at this point and reinvest the total amount and then we calculate what is going to be the total amount of money at this point
 so if the total amount of interest till this point is one in the half of the period the interest rate is going to be half
 so x amount of money at this point is going to be x into 1 plus half and this amount of money reinvested here
 so at the end of this time period it is going to be x into 1 plus half whole square this you must have seen in calculation of compound interest now let us increase the time period further suppose we decide to give one third interest at this point one third interest here and one third here therefore by the same logic x amount of money at this point is going to be x into 1 plus 1 upon 3 which at this point is going to be x into 1 plus 1 upon 3 whole square which at this point is going to be x into 1 plus 1 upon 3 whole cube thus you can

understand that as i am divide making more partitions and paying the interest at each of this time point then the amount that we receive is different

so what is going to be the amount if i partition it at in time periods by the same logic we can see that this money is going to be x into 1 plus 1 upon n whole to the power n

so that gives you an idea where from these terms come 1 plus 1 upon n whole to the power n

so as n increases with time shall we get infinite amount of money that will depend upon where does it converge

so i wanted to calculate for your benefit the value of this at some end points for example at n is equal to two it is one plus zero point five whole square which is is equal to two point two five at n is equal to ten this is one plus zero point one whole to the power ten which is going to be roughly 2.594 at n is equal to 100 we get 1 plus zero point zero one whole to the power hundred which is going to be two point seven zero 4 8 if you have access to some scientific calculator you can compute these and go up to 4 decimal place you will find that the values are coming out to be like this at n is equal to thousand it is one plus zero point zero zero one whole to the power thousand which is going to be two point seven one six nine at n is equal to ten thousand it is one plus zero point zero zero zero one whole to the power ten thousand which is is equal to two point seven one eight one and at n is equal to one lakh it is one plus zero point zero zero zero zero one to the power one lakh which is going to be two point seven one eight three

so we can see that the values are increasing but not at a very high rate in fact as n goes to infinity one plus one upon n whole to the power n will converge to a constant which is two point seven one eight two eight one eight 2 8 4 5 9 0 4 5 there are there is no end as this is an irrational number will never come to an end of this sequence and people have tried to compute up to 1000 decimal places there was no convergence

so this irrational number is called euler's number and which is denoted by e

so so let us consider 1 plus 1 by n whole to the power n what is going to be the kth term we can see that it is a binomial expansion

so for a given k if we consider n greater than k then the then the coefficient is going to be n into n minus 1 into n minus k minus 1 upon it is nck

so it is factorial k or in other words into 1 by n whole to the power k

so this is going to be the kth term we get it from nck into 1 by n whole to the power k now if we cancel out this n we get it as 1 into 1 minus 1 by n into 1 minus 2 by n 1 minus k minus 1 by n upon factorial k therefore as n goes to infinity the kth term is one upon factorial k therefore the series 1 plus 1 by n whole to the power n goes to 1 plus 1 plus 1 upon factorial 2 plus 1 upon factorial three plus one upon factorial k in the next class i shall solve some problems involving the euler constant and its expansion thank you

so much you