welcome students to the third lecture on infinite series in the last lecture i was talking about binomial expansions of the form one minus x whole to the power minus n n is an integer or one minus x to the power p by q which is a rational number in particular we were looking at one minus x whole to the power minus half and we have seen the coefficients of its expansion for different powers of x let us take a simpler problem what is the series expansion for one minus x to the power half

so this is a rational number let us see how to go about it we have one minus x whole to the power half into 1 minus x whole to the power half is equal to 1 minus x therefore if we write one minus x whole to the power half is equal to a zero plus a one x plus a two x square etcetera etcetera then by multiplying it with itself we should get one minus x

so let us try that therefore a 0 plus a 1 x plus a 2 x square plus multiplied by a 0 plus a 1 x plus a two x square is equal to one minus x therefore coeff of coefficient of x to the power zero is equal to a zero square is equal to one again taking the positive root a zero is equal to one coefficient of x is equal to a zero a one plus a one a zero is equal to two a zero a one which is is equal to minus one therefore 2 a 1 is equal to minus 1 therefore a 1 is equal to minus half coefficient of x square is equal to a 0 a 2 plus a 1 square plus a 2 a 0 is equal to 0 y 0 because there is no x square in 1 minus x therefore a zero is equal to one we have already got therefore two a two plus a one square is equal to zero or two a two plus a one is equal to minus half

so u square is equal to one by four is equal to zero therefore a two is equal to minus one by eight or we can write 1 minus x to the power half is equal to square root of 1 minus x as 1 minus half x minus 1 by 8 x square plus higher powers of x we are going only up to the second degree of x but if x is small we often ignore higher powers ok example what is consider root over seventeen we can write in a similar way we can find root over one plus x say root over one plus x is equal to a zero plus a one x plus a two x square etcetera therefore in a similar way we can write that a zero plus a one x plus a two x square into a zero plus a one x plus a two x square plus is equal to one plus x now by multiplying them and equating the powers of x we find a 0 square is equal to 1 or a zero is equal to one by taking positive value a zero a one plus a 1 a 0 is equal to 1 or 2 a 0 a 1 is equal to 1 therefore a one is equal to half a zero a two plus a one square plus a zero a two is equal to 0 or 2 a 2 plus a 1 square is equal to 0 therefore 2 a 2 is equal to minus a one square is equal to minus one by four therefore a two is equal to minus one by eight therefore we get square root of one plus x is equal to one plus half x minus 1 by 8 x square plus other terms which we are ignoring for the time being let us apply this find root over 17 suppose we need to find out the square root of 17 we can write it as but if we write it like that we are making a mistake what is the mistake because this x in 1 plus x whole to the power some p in the expansion modulus of x has to be less than one but if we write it like one plus sixteen whole to the power half then we are making a mistake

so we write it in a different way we write it as 16 power half into 1 plus 1 by 16 whole to the power half right

so that gives us a term 1 by 16 whose mod value is less than 1 therefore to obtain root 17 we write it as 16 to the power half into 1 plus 1 upon 16 whole to the power half is equal to 4 into 1 plus 1 by 16 whole to the power half and now let us expand it using binomial we have found out that in the expansion of one plus x whole to the power half we have found out this is is equal to one plus half x minus one by eight x square plus other terms which we have ignored putting x is equal to 1 by 16 we get this is is equal to 1 plus 1 by 32 minus 1 by 8 into 16 square therefore root 17 is equal to 4 into 1 plus 1 by 32 minus 1

by 8 into 16 square
 so 4 into 1 is equal to 4 4 into 1 upon 32 is equal to 1 upon 8 is equal to zero point one two five four into minus one upon eight into sixteen square is equal to minus one upon two into sixteen square is equal to two into two hundred fifty six is equal to one upon five hundred and twelve is equal to minus zero point zero zero one nine therefore square root of 17 is equal to 4.
125 minus 0.
0019 is equal to four point one two three one now if we consider route 17 i suggest all of you use your calculator to compute root 17 and you see it comes
 so close to four point one two three one
 so this is a verification that this expansion works in a correct manner therefore the final expansion is 1 plus 6 whole to the power p by q we will write in a very similar way as we did with respect to negative integral index this is is equal to 1 plus p by q to the power x plus p by q into p by q minus 1 upon factorial 2 to the power x square plus p by q into p by q minus 1 into p by q minus 2 whole to the power factorial 3 to the power x cube plus therefore irrespective of when we have a binomial expansion of the form 1 plus x whole to the power sum index irrespective of whether it is positive integral negative integral or a rational we can write it in the same way and only thing that we have to remember that for positive integral we can write it as n choose r or n c r that we cannot do when we have a negative integral index or a rational index like p by q but we can rewrite it in the following form and we can get the series expansion for any binomial expression with power as an integer or a rational number again this is not a proof right what we have done
 so far is not a proof we have just verified certain results and the assumption as i said in the very first class is y stars approximation theorem which suggest that every continuous function in a closed interval can be approximated as can be approximated as closely as possible by a polynomial function therefore what we have done given a continuous function we have tried to compute the coefficients of first few powers of x and thereby we try to find the expansion of one plus x whole to the power k where k can be negative integral or rational let us see how to obtain the coefficients how to obtain the coefficients we make such an assumption that given a function f of x it is possible to have such an such a polynomial expansion about a point a
 so let us write f of x is equal to a 0 plus a 1 into x minus a plus a 2 into x minus a whole square plus a three into x minus a whole cube etcetera the advantage of polynomial is that if it is of degree n it can be differentiated n plus one times and if we take an infinite polynomial then we can differentiate it in finite number of times
 so with this assumption we shall try to find out the polynomial expansion for f x f x is equal to we have assumed as a 0 plus a 1 into x minus a plus a 2 into x minus a whole square plus a 3 into x minus a whole cube etcetera therefore f at a is equal to a 0 because all other terms will become 0 therefore a 0 is equal to f a thus the constant term will come the functional value at a point a about which we are expanding the polynomial what is the first derivative of a
 so i am writing it as f of upper script one which means i am differentiating f with respect to x once this is is equal to a 1 plus 2 a 2 into x minus a plus 3 a 3 into x minus a whole square plus 4 a 4 x minus a whole cube like that therefore what is the second derivative second derivative of x is two a two plus three into two a three into x minus a plus 4 into 3 a 4 into x minus a whole square etcetera therefore if two at a is equal to twice a two or a two is equal to f two at a by two in a similar way the third derivative f three of x is equal to 3 into 2 into 1 a 3 plus 4 into 3 into 2 x minus a plus terms with higher powers of x minus a therefore if three at a is equal to three factorial times a

three therefore a three is equal to f three a upon three factorial in a similar way we can see that if i differentiate it once more i will get the term f 4 at x is equal to 4 into three into two into one plus powers of x minus a therefore a four is equal to fourth derivative of x at a divided by factorial four therefore we can find that f of x can be written as f at a plus f prime at a into x minus a plus f second derivative of f at a into x minus a whole square upon factorial two plus third derivative of f at a into x minus a whole cube upon factorial three plus fourth derivative of a into x minus a whole to the power four upon factorial four it is not mandatory that you have to go to infinity we can always make an approximation by expanding it to a fixed power say k is equal to 4 and then the remaining term will be the error term in the approximation but if the difference between x and a is very small then as the power increases then the error term will go to zero

so this expansion is called taylor series expansion of f x when f is continuous and differentiable in finite number of times at the point a you shall stand study more about taylor series in your higher classes of mathematics but in this class we shall see how it helps us in solving certain problems let us consider 1 minus x whole to the power minus 2 therefore f x is equal to 1 minus x to the power minus 2 f first derivative of x is equal to minus 2 into 1 minus x to the power minus 3 into minus 1 is equal to 2 into 1 minus x to the power minus 3 second derivative of x is equal to minus 3 into 2 into 1 minus x to the power minus 4 is equal to factorial 3 into one minus x to the power minus four f third derivative of x is equal to in a similar way factorial 4 into 1 minus x whole to the power minus 5 therefore if at zero is equal to one a prime at zero is equal to two if second derivative at zero is equal to factorial three and if fourth derivative at zero is equal to factorial four therefore we get f of x is equal to f at zero plus two into x minus 0 plus factorial 3 into x minus 0 whole square upon factorial two plus factorial four into x minus zero whole cube upon factorial three therefore we can write f x is equal to f at a plus if first derivative at a into x minus a plus if second derivative at a into x minus a whole square upon factorial 2 plus f third derivative at a in x minus a whole cube upon factorial 3 plus a 4th derivative of a into x minus a whole to the power four upon factorial four like that and putting the values f of x is equal to one minus x whole to the power minus two is equal to f at 0 plus f prime at 0 into x plus if second derivate at zero into x square upon factorial two plus third derivative of f at zero into x cube upon factorial 3 plus 4th derivative of f at 0 into x 4 upon factorial 4 is equal to f 0 plus 2 times x plus factorial three into x minus zero whole square upon factorial two plus factorial four into x minus zero whole cube upon factorial three etcetera is equal to one plus two x plus three x square plus 4 x cube plus and these are the terms that we have already seen right we have already seen that 1 minus x whole to the power minus two is equal to one plus two x plus three x square plus four x cube like that

so taylor series expansion works for 1 minus x whole to the power minus 2 i want you to verify the same with other polynomial expansions that we have done already in this class and i shall go beyond polynomials if i ask you what are the other functions that you can remember very easily which is differentiable smoothly for different values of x the first one that comes to my mind is the trigonometric functions in particular let us look at sine x cos x tan x right let us see whether we can expand this using taylor series expansion and find a way to compute sine of x or cos of x or tan of x etcetera because if you remember in classes we see the values of sine x cos x etcetera are only for a fixed set of values right we have seen for zero degree for pi by six for pi by four pi by three pi by two and pi and typically we work with their multiples or

perhaps with some more trigonometric manipulations we can get for 15 degrees 18 degrees etcetera right what if i ask you what is the sign of one degree or what is the sign of five degrees it is not easy to compute those values unless we use a taylor series expansion that is why this formula is very very important for analysis

so for illustration consider sine x f x is equal to sine x therefore f zero is equal to zero first derivative of x is equal to cos x therefore first derivative at zero is equal to cos zero which is is equal to one second derivative of x is equal to minus sin x therefore second derivative at zero is equal to zero third derivative of x is equal to minus cos x therefore third derivative at zero is equal to minus one i will go for a few more the fourth derivative of x is equal to sine x therefore fourth derivative at zero is equal to zero fifth derivative of x is equal to cos x therefore fifth derivative at zero is equal to one

so let us stop here and we can we know that from taylor's theorem from taylor's series f of x is equal to f of 0 plus f prime at 0 into x minus 0 plus f second derivative at 0 into x minus a whole square upon factorial two etcetera therefore from taylor series f of x is equal to f at zero plus f prime at 0 into x minus 0 plus f second derivative at 0 into x minus a whole square upon factorial 2 plus derivative at 0 into x minus a whole cube upon factorial 3 plus 4th derivative of 0 into x minus a whole to the power 4 upon factorial four plus a fifth derivative of zero into x minus a whole five upon factorial five plus like that we will go i am not going any further let us replace the values f zero is equal to zero f first derivative at zero is equal to one if second derivative at zero is equal to zero if third derivative at zero is equal to minus one a fourth derivative at zero is equal to zero and a fifth derivative at zero is equal to minus is equal to plus one this is what we have got putting these values we get sine of x is equal to sine at 0 plus 1 times x minus 0 plus 0 times x minus 0 whole square upon factorial 2 minus 1 times x minus 0 whole cube upon factorial 3 plus 0 times x minus 0 whole to the power 4 upon factorial 4 plus one times x minus zero whole five upon factorial five like that this is is equal to one times x minus x cube upon factorial three plus x to the power five upon factorial five if we continue further we shall see that this is is equal to x minus x cube upon factorial 3 plus x to the power 5 upon factorial 5 minus x to the power 7 upon factorial 7 like that that is we shall get a polynomial where only all powers of x are there and the coefficient for x to the power k is one upon factorial k and the signs of them are going to be plus minus plus minus in an alternative way similarly we can work out for cos x and i like you to verify that cos x is equal to 1 minus x square upon factorial 2 plus x to the power 4 upon factorial 4 minus x to the power 6 upon factorial 6 like that that is when we look at cos x we find only even powers of x and again like sign we get alternatively positive and negative signs next we look at tan x again we expand about 0 f 0 is equal to 10 0 is equal to zero if one at x is equal to d d x of tan x is equal to 6 square x is equal to 1 plus tan square x therefore f one at zero is equal to one what about f two x that is second derivative of x is equal to d d x of 1 plus tan square x is equal to 2 tan x into six square x is equal to two tan x into one plus tan square x is equal to 2 tan x plus 2 tan cube x right therefore f 2 at 0 is again 0 but we needed this expansion

so that we can go to the third derivative therefore f 3 x is equal to 2 into 1 plus tan square x plus 6 tan square x into 1 plus tan square x is equal to 2 plus 2 tan square x plus 6 tan square x plus 6 tan 4 x therefore f 3 at 0 is equal to two if four x is equal to d d x of this because derivative of the constant is 0 the over x 4 x is equal to d d f of a 8 tan square x plus 6 tan 4 10 to the power 4 x is equal to 16 tan x plus 16 tan cube x plus 24 tan cube x into 1 plus tan square x therefore f 4 at 0 is equal to zero in a similar way we

can get if five x is equal to 16 into 1 plus tan square x plus terms with tan x therefore f five at zero is equal to sixteen therefore by expanding up to this point we can get that tan x is equal to zero plus one upon one factorial x plus 0 time x square upon factorial 2 plus 2 times x cube upon factorial 3 plus 0 times x 4 upon factorial 4 plus 16 times x to the power 5 upon factorial 5 plus higher powers of x on simplification we get tan of x is equal to 0 plus x plus 0 times x square plus 2 into x cube upon factorial 3 plus 0 times x to the power 4 plus sixteen times x to the power five upon factorial five now this is is equal to x plus x cube upon three plus sixteen upon factorial five is equal to one twenty

 so this is four upon thirty is equal to two upon fifteen is equal to two upon fifteen times x to the power five plus higher powers of thus we get an approximation of tan x in the form of a polynomial up to fifth power of x okay students i stop here today in the next class i look at some more problems in particular i will look at how to obtain the taylor series expansion for tan inverse x for logarithmic functions in particular log of 1 plus x and more importantly the expansion of e to the power x which is very important in analysis okay you