

welcome students to the second lecture on infinite series if you remember in the first lecture we are looking at binomial expansions of expressions such as  $(1+r)^n$  which is equal to  $1 + r + r^2 + r^3 + \dots$  that is  $\sum_{i=0}^{\infty} r^i$  and this is valid for modulus of  $r$  less than one also we have seen that  $(1-r)^n$  is equal to  $1 - r + r^2 - r^3 + \dots$  this is a finite series which is equal to  $\sum_{i=0}^{\infty} (-1)^i r^i$  again modulus of  $r$  is less than one today we will start with some simple problem to verify these expressions ok

so problem one what is zero point eight inverse we all know that zero point eight is equal to eight by ten and its inverse means we are looking at ten by eight which we all of us know that is one point two five

so in this case we know the answer

so what i like to show is that can we get the same answer by expanding it in finite

so our question is can we get the same answer by expanding its infinite sum how to do that

so we look at it in the following way we know zero point eight inverse is equal to one minus zero point two inverse

so this comes of the form  $(1-r)^{-1}$  where modulus of  $r$  is less than one since zero point two modulus is less than one

so this condition is satisfied therefore we can expand it with the series expansion that we have seen at the beginning

so let us try whether it holds good or not therefore by the series expansion this is  $1 + 0.2 + 0.2^2 + 0.2^3 + \dots$

$1 + 0.2 + 0.2^2 + 0.2^3 + \dots$

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$1 + 0.2 + 0.2^2 + 0.2^3 + \dots$  if we consider this part we have already got one point two but we know the answer is one point two five

so our target is to show whether this sum is equal to zero point zero five or not

so what is this sum this sum is  $0.2 + 0.2^2 + 0.2^3 + \dots$

$0.2 + 0.2^2 + 0.2^3 + \dots$

$0.2 + 0.2^2 + 0.2^3 + \dots$  plus zero point two whole to the power four plus this infinite sum which is equal to if we take out zero point two square common  $0.2 + 0.2^2 + 0.2^3 + \dots$

$0.2 + 0.2^2 + 0.2^3 + \dots$

$0.2 + 0.2^2 + 0.2^3 + \dots$  and this is a gp series therefore this is equal to  $0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  and we know that this sum is  $(1-r)^{-1}$  where  $r = 0.2$  which is equal to  $0.2$  multiplied by  $(1-0.2)^{-1}$

$0.2 + 0.2^2 + 0.2^3 + \dots$  whole to the power  $n$  upon  $1 - 0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  and take limit  $n$  goes to infinity therefore this becomes  $0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  multiplied by  $1$  upon  $1 - 0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  is equal to  $0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  into  $1$  upon  $0.8$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  which is equal to  $0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  upon  $0.8$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  equal to  $1$  upon  $20$  which is equal to  $0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  therefore the whole sum is  $1$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  plus  $0.2$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  equal to  $1$ .

$0.2 + 0.2^2 + 0.2^3 + \dots$  and we know that that is the right answer

so this example suggest that we can get the right answer by expanding the

binomial series the way we have done one minus  $r$  whole to the power minus one is equal to one plus  $r$  plus  $r$  square plus  $r$  cube up to infinity let us take a very similar problem problem number two what is one plus zero point two to the power minus one we know the answer we know the answer because this is 1.

$2^{-1}$  is equal to  $10^{-1}$  upon  $10$  inverse which is is equal to  $10$  upon  $12$  which is is equal to  $5$  upon  $6$  which is is equal to zero point eight three three three like that okay

so we know the answer we shall have to see whether we get the same answer by expanding it as a series we have the formula  $1 + r$  whole to the power minus one is equal to  $1 - r + r^2 - r^3$  etcetera therefore  $1 + 0$ .

$2^{-1}$  is  $1 - 0$ .

$2^2$  plus  $0$ .

$2^2$  square minus  $0$ .

$2^3$  cube plus like that we have already got here  $0$ .

8 therefore what needs to be checked is whether this is going to be zero point zero three three three all right

so this is what we need to check and like the previous problem we can look at it as a geometric series it is  $0$ .

$2^2$  whole square into  $1 - 0$ .

$2^2$  plus  $0$ .

$2^2$  square minus like that

so this is the infinite series where common ratio is minus  $0$ .

$2$  therefore this sum is zero point zero four multiplied by one upon one plus zero point two this is  $0$ .

$04$  multiplied by  $1$  upon one point two which is is equal to four upon one twenty which is is equal to one upon thirty which is is equal to zero point zero three three three three therefore the answer is zero point eight three three three therefore we see that for both one minus  $r$  whole to the power minus one and one plus  $r$  whole to the power minus one we can verify that the series actually gives the right answer when we check it with some known values note that

so far we have not really proved it but we have just verified now the question is is it true for only minus one can we have similar expansion say for minus two minus three or what about some rational number say half two by three that is the question in short we have earlier studied binomial theorem for positive integer  $n$  and we know that  $1 + x$  whole to the power  $n$  is equal to  $1 + n C_1 x + n C_2 x^2 + \dots + n C_{n-1} x^{n-1} + x^n$

so the characteristic that we had for positive integral index  $n$  is that number of terms is finite  $b$  we could use combination  $n C_r$  for  $r$  is equal to  $0$   $1$  up to  $n$  we could do these things the problem with negative integral negative index is one the number of terms is infinite but more importantly we cannot use minus  $n C_r$  as this is not defined therefore we have to do things in a slightly different way but before that let us first see what is one minus  $r$  whole to the power minus two again we assume that we can write it as a polynomial in  $r$  which will be convergent for modulus of  $r$  less than one our aim is to find out the coefficients of the expansion right if this is not finite if this is finite then what will happen then after certain end the corresponding coefficients will become zero

so let us target to find out the coefficients of one minus  $r$  whole to the power minus two let us assume one minus  $r$  whole to the power minus two is equal to a zero plus  $a_1 r$  plus  $a_2 r^2$  like that ok our aim is to find these coefficients we know that one minus  $r$  whole to the power minus two is equal to one minus  $r$  whole to the power minus one multiplied by one minus  $r$  to the power minus one therefore we can write this as the product of the corresponding series which is is equal to  $1 + r + r^2 + \dots$  multiplied by  $1 + r + r^2 + \dots$

square plus then what we will try to do we will try to find out coefficients of  $r$  to the power  $k$  for different powers of  $k$  and then we will try to compare a  $0$  a  $1$  a  $2$  with those values

so let us write it again  $1 + r + r^2$  multiplied by  $1 + r + r^2$  therefore coefficient of  $r$  to the power  $0$  we can see that the only term that does not involve any  $r$  is product of this into this is equal to one therefore a zero is equal to one coefficient of  $r$  is equal to we can get  $r$  in two ways this one multiplied with this  $r$  and these are multiplied with this one therefore coefficient of  $r$  is equal to one plus one is equal to two therefore a one is equal to two what is coefficient of  $r^2$  we can find  $r^2$  in three different ways we can compute  $r^2$  in three different ways one into  $r^2$  plus  $r$  into  $r$  plus  $r^2$  into  $1$  because other terms are higher powers of  $r$  none of them will contribute to  $r^2$  therefore coefficient of  $r^2$  is  $1 + 1 + 1$  is equal to  $3$  therefore a  $2$  is equal to  $3$ .

let me go for one more term that will make you understand the pattern coefficient of  $r^3$  in a similar way  $1 + r + r^2 + r^3$  into  $r + r^2 + r^3$  into  $1$  and then  $r^3$  into  $1$  all four of them will give  $r^3$  therefore coefficient of  $r^3$  is equal to four therefore a three is equal to four in general coefficient of  $r$  to the power  $k$  is equal to  $k + 1$  right because there will be  $k + 1$  terms  $1 + r + r^2 + \dots + r^k$  each one of them when multiplied with one particular term here will give you  $r^k$  therefore we can write  $1 - r$  whole to the power  $n - 1$  is equal to  $1 + r + r^2 + \dots + r^{n-1}$  like that we can get an infinite sum for the expression  $1 - r$  whole to the power  $n - 1$  from here we can easily see that  $1 - r$  whole to the power  $n - 1$  is equal to  $1 - r^n$  therefore this is going to be  $1 - r^n$  plus  $r^n$  minus  $r^{2n}$  that I suggest that you expand  $1 - r$  whole to the power  $n - 1$  as a product of  $1 - r$  to the power  $n - 1$  into  $1 + r + r^2 + \dots + r^{n-1}$  and then try to match the coefficients and try to obtain this as the expression for  $1 - r$  whole to the power  $n - 1$  let me go to the next problem what is  $1 - r$  whole to the power  $n - 2$  again as before we assume  $1 - r$  whole to the power  $n - 2$  is equal to that in finite series  $b_0 + b_1 r + b_2 r^2 + b_3 r^3 + \dots$  and we try to find the values of  $b_0, b_1, b_2, \dots$  how to do that we will do that in the following way we will write  $1 - r$  whole to the power  $n - 2$  is equal to  $1 - r$  to the power  $n - 2$  into  $1 - r$  to the power  $n - 1$  we already know the series expansion for this we already know the series expansion for this therefore the way we have done for  $1 - r$  whole to the power  $n - 1$  we will try to do in a similar way for  $1 - r$  whole to the power  $n - 2$  and  $1 - r$  to the power  $n - 1$  multiplied by  $1 - r$  to the power  $n - 1$  is equal to  $1 + 2r + 3r^2 + 4r^3 + \dots$  like that multiplied by  $1 + r + r^2 + r^3 + \dots$

so in the product coefficient of  $r$  to the power  $0$  is equal to  $1$  multiplied by  $1$  is equal to  $1$  therefore  $b_0$  is equal to one coefficient of  $r$  is equal to one times this  $r$  plus two  $r$  times one can you see that this one multiplied with  $r$  will give one  $r$  and these two are multiplied with one with this one will give two

so this is one plus two whole to the power one plus two therefore  $b_1$  is equal to three coefficient of  $r^2$  now you can easily find out this one multiplied by this  $r^2$  will give me one plus these two are multiplied with this  $r$  will give me two  $r^2$  plus this three  $r^2$  multiplied with this one will give me three  $r^2$  therefore it is six therefore  $b_2$  is equal to six I will go for one more term coefficient of  $r^3$  is equal to one multiplied by  $r^3$  that is one plus two  $r$  multiplied by  $r^2$  that is two plus three  $r$

multiplied by  $r$  that is three plus four  $r$  multiplied by one four  $r$  cube multiplied by one that is four is equal to ten therefore  $b^3$  is equal to ten so this gives us an idea of the series therefore what is going to be the general term is 1 multiplied by  $k+1$  plus 1 multiplied by  $k+1$  multiplied by  $k-1$  up to one multiplied with one which is is equal to 1 multiplied by 1 plus 2 plus 3 up to  $k+1$  which is is equal to  $k+1$  into  $k+2$  by two this is going to be  $b^k$  is equal to coefficient of  $r$  to the power  $k$

so you look at it when  $r$  is equal to zero it is one into two by two this is one when  $k$  is equal to one it is two plus three by two is equal to three when  $k$  is equal to two it is three plus four three multiplied by four which is twelve divided by two is equal to six when  $k$  is equal to three it is four multiplied by five is equal to twenty divided by two is equal to ten therefore we get an expression for one minus  $r$  whole to the power minus three as follows one minus  $r$  whole to the power minus 3 is therefore  $\sum_{k=0}^{\infty} \binom{-3}{k} r^k$  obviously it is not possible to compute these terms for arbitrarily large  $n$  we need to derive a formula what is that for positive  $n$  for positive integral  $n$  we have coefficient of  $r$  to the power  $k$  is equal to  $\binom{n}{k}$  we have already seen that such terms are not valid for negative  $n$  what we do we write  $\binom{n}{r}$  as factorial  $n$  upon factorial  $r$  into factorial  $n-r$  is equal to we can now cancel few terms from  $n$  factorial and  $n-r$  factorial so what we get is  $n$  into  $n-1$  up to  $n-r$  divided by  $r$  factorial

so this gives us the formula

so let us denote by  $c_{-n, k}$  the term  $\binom{-n}{k}$  minus  $n$  into minus  $n-1$  into minus  $n-2$  up to minus  $n-k+1$  divided by  $k$  factorial we shall see that  $c_{-n, k}$  is the coefficient of  $r$  to the power  $k$  in the expansion of one plus  $x$  whole to the power minus  $n$  where modulus of  $x$  is less than one verification consider one minus  $r$  whole to the power minus two coefficient of  $r$  is equal to minus two upon one factorial is equal to coefficient of minus  $r$  is equal to minus two one factorial is equal to minus 2 coefficient of minus  $r$  square is equal to minus 2 minus 3 upon 2 factorial which is is equal to 3 coefficient of minus  $r$  cube is equal to minus 2 minus 3 minus 4 upon factorial 3 which is is equal to 4 like that minus 2 minus 3 minus 4 upon factorial 3 which is is equal to minus 4 like that therefore the expansion is 1 plus minus 2 into minus  $r$  plus 3 into minus  $r$  whole square plus minus 4 into minus  $r$  whole cube like that is equal to one plus two  $r$  plus three  $r$  square plus four  $r$  cube or we see that this is the series that we have got when we have made the algebraic manipulation

so this is one verification of the result that i have just stated let us verify for 1 plus  $r$  whole to the power minus 3 coeff of  $r$  is equal to minus 3 upon 1 factorial which is is equal to minus 3 coefficient of  $r$  square is equal to minus 3 into minus 4 upon 2 factorial which is is equal to plus 6 coefficient of  $r$  cube is equal to minus 3 minus 4 minus 5 upon factorial 3 which is is equal to minus 10 like that therefore we can see that 1 plus  $r$  whole to the power minus 3 is equal to 1 minus 3  $r$  plus 6  $r$  square minus 10  $r$  cube and we have seen the similar result for one minus  $r$  whole to the power minus three putting  $r$  in place of minus  $r$  will get this result next we shall look into binomial expansion with fractional index that is what is 1 minus  $x$  whole to the power half or 1 minus  $x$  whole to the power minus half let us try consider one minus  $x$  whole to the power minus half we know that 1 minus 6 whole to the power minus 1 is equal to 1 minus  $x$  whole to the power minus half into 1 minus  $x$  whole to the power minus half this is is equal to 1 plus  $x$  plus  $x$  square plus  $x$  cube for modulus of  $x$  less than one let us assume one minus  $x$  whole to the power minus half is equal to a  $\theta$

plus a 1 x plus a 2 x square like that then we can multiply this by itself and if we equate the coefficients with 1 plus x plus x square plus x cube this series then we will get different equations and from there we will try to solve these values for a 0 a 1 a 2 etcetera

so we have is equal to a zero plus a one x plus a two x square multiplied by a zero plus a one x plus a two x square let me write one more term for each is equal to 1 plus x plus x square plus x cube therefore what we get the constant is a 0 square and that we can equate with the constant of this series which is one therefore a zero square is equal to one therefore a 0 is equal to plus minus 1 but we take the positive factor and c therefore let us assume that a 0 is equal to 1 what is the coefficient of x it is a 0 a 1 plus a 1 a 0 is equal to 2 into a 0 into a 1 is equal to 2 a one which is is equal to one this comes from the series one minus x whole to the power minus one therefore two a one is equal to one therefore a one is equal to half coefficient of x square is equal to a 0 into a 2 plus a 1 square plus a 2 into a 0 is equal to one therefore two a zero a two plus a one square is equal to one therefore two a zero a two is equal to one minus one by four is equal to three by four therefore two a two is equal to three by four therefore a two is equal to three by eight therefore we get that one minus x whole to the power minus half is equal to 1 plus half into x plus 3 by 8 into x square plus dotted we will see whether we get the same by using the expansion the way we did for negative integers

so for negative integers what we did if you remember we have seen that coefficient of x to the power k in 1 minus x whole to the power minus n is equal to minus n minus n minus 1 minus n minus k plus 1 upon factorial k let us apply the same for p by q in particular here for minus half therefore by using similar expansion we get 1 minus x whole to the power minus half is equal to 1 minus minus half into x plus minus half into minus half minus 1 upon factorial 2 square etcetera is equal to 1 plus half x plus 3 by 8 x square and we have already obtained that a zero is equal to one a one is equal to half a two is equal to three by eight thus we get the same answer okay friends with that i stop this session in the next session i shall expand more with this formula thank you you