

welcome students to the first lecture of infinite series in this and subsequent few lectures i will be talking about series with infinite number of terms this is a very important concept in analysis as you will find that we can compute values for many functions for a given specific value of  $x$  say for example  $\sin x$   $\cos x$  for arbitrary value of  $x$  we can compute using corresponding finite series say for example  $\sin x$  you have

so far seen only for a few values right say for zero  $\pi$  by three  $\pi$  by four  $\pi$  by six or  $\pi$  by two something like that but actually  $\sin$  is a continuous function and if you have seen the  $\sin$  curve it is defined for all the different values of  $x$  on the real line question comes how do you compute the value of  $\sin x$  for any arbitrary given value of  $x$

so this is true not only for  $\sin x$  many such functions are there and therefore infinite series gives us the way to compute such values the basics of infinite series for computing functional values comes from Weierstrass approximation theorem which suggests that if  $f$  is a continuous real valued function defined on the closed interval then for every real  $x$  belonging to the interval  $a$  to  $b$  and given any  $\epsilon$  greater than zero there exists a polynomial  $p$  such that for all  $x$  belonging to  $a$  comma  $b$  modulus of  $f(x) - p(x)$  is less than  $\epsilon$

so this is very important

so  $f$  is a continuous function in  $a$  to  $b$  suppose this is the real line and this is  $a$  and this is  $b$  and  $f$  is arbitrary continuous function then at any particular value  $x$  i can approximate  $f(x)$  with a polynomial and you all know that polynomial of degree  $n$  is of the form say  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

so depending upon the value of  $\epsilon$  we can choose a large  $n$

so that value of the polynomial will be close enough of the functional value  $f(x)$  in other words we can approximate  $f(x)$  by a polynomial  $p(x)$

so if for an arbitrary function say  $\sin x$  if we can find a polynomial function to approximate  $\sin x$  then we are done the question is how to obtain such a polynomial in order to do that we have to look at a series now what is a series in the iit pal lectures we have dealt with sequence and series look at these lectures you will get a foundation of sequence and series but to explain the concept of infinite series i will give you a small introduction to these two terms sequence and series and some examples

so that you can recapitulate those things easily ok

so what is the sequence a sequence is an ordered arrangement of certain entities or objects for example alphabet if we consider english alphabet we have a b c up to z these 26 letters come in that specific order right nobody says c z b m no there is an order and that is the sequence in which the different letters occur in the english alphabet similarly say rainbow colors there is also a sequence violet indigo blue green yellow orange and red now a sequence can be numeric also say prime numbers what are the prime numbers 2 3 5 7 11 13.

so there is a specific sequence in which the prime numbers are occurring even numbers if you look at them they are two four six eight ten now we can see that two things we have to observe firstly sequence may be finite or infinite right alphabet colors of rainbow etcetera are finite sequence what does it mean that is the number of terms is finite even numerically we can think of say all odd numbers between 0 to 100 and we know there are only finite number of odd numbers namely one three five up to ninety nine finite sequences are somewhat easy to handle we can easily find their maximum minimum sum average etcetera but when it comes to infinite series in infinite sequence that means the number of terms can be arbitrarily large what does it mean it means that you choose any large number in the number of terms in the sequence is greater than  $n$  that means you can find more than  $n$  number of elements of that sequence example all even numbers all

powers of two say for example two four eight sixteen thirty two you can keep on increasing the values which are powers of two that is given any  $k$   $2$  to the power  $k$  belongs to the above sequence the second issue is whether the sequence has a limit consider for example one two three sequence of natural numbers does it have a limit you all know that the answer is no you can give me any number capital  $n$   $i$  will show that  $i$  can find an integer that is bigger than that similarly  $2$  to the power  $k$   $k$  is equal to  $1$   $2$  etcetera does not have any limit as you give me any big number  $n$   $i$  can always find the  $k$  such that  $2$  to the power  $k$  is bigger than that  $n$  on the other hand considered  $1$  plus  $1$  by  $k$   $k$  is equal to  $1$   $2$   $3$  etcetera what will happen at  $k$  is equal to one the value is the two  $k$  is equal to two one plus half is equal to one point five at  $k$  is equal to ten as you can understand it is going to be one point one  $k$  is equal to hundred it is going to be one point zero one  $k$  is equal to thousand one point zero zero one what do  $i$  find that it will always be greater than one but as  $k$  increases it is close to  $1$  right as  $k$  increases the values are coming close and close to one a different type of example is the following consider minus one whole to the power  $n$  if we write the sequence it will look like the following minus one power one next one is minus one power two next one is minus one power three next one is minus one power four and

so on and that is equal to minus one plus one minus one plus one

so we can see alternate occurrence of minus one and plus one

so as  $n$  increases the values of the sequence are not converging to any value  $r$  right therefore this sequence also does not have any limit now the next term that  $i$  like to explain is a series you all know that given a sequence say  $a_1$   $a_2$   $a_3$  like that where  $a_k$  is the  $k$ th term of the sequence the term the summation  $a_1$  plus  $a_2$  plus  $a_3$  or  $\sum_{k=1}^n a_k$  say  $\sum_{k=1}^n a_k$   $k$  is equal to  $1$   $2$   $3$  etcetera is called the series corresponding to the given sequence  $a_1$   $a_2$   $a_3$  etcetera now if the sequence is finite and the terms are numeric then we can easily obtain the value of the series or sum of the series in fact you have seen a  $p$   $j$   $g$   $p$  etcetera and we have obtained the sum of the first  $10$  terms right

so we know if a given series is in  $ap$  then we know how to obtain the first  $n$  terms if the given series is in  $gp$  or geometric progression we know how to obtain the sum of first  $n$  terms question is what happens if the sequence is infinite that means the number of terms in that sequence is infinite e  $g$   $a_k$  is equal to  $2$  to the power  $k$  another example  $a_k$  is equal to  $5$  minus  $2^k$  another example  $a_k$  is equal to  $\sin$  of  $2^k \pi$  another example  $a_k$  is equal to  $\sin$  of one upon two  $k \pi$  by two or one upon  $k \pi$  by two if you look at it in each one of them there are infinitely many terms right you can always substitute the value of  $k$  in the formula for the corresponding sequence and we can obtain the value of the  $k$ th term

so if we add them we get a series  $\sum_{k=1}^{\infty} a_k$  the question comes whether the series will converge to some particular value or not that is the question is if the series is in finite does it converge to some value or does it have a limiting value

so we do it in the following way let  $s_n = \sum_{k=1}^n a_k$  for  $k$  is equal to one to  $n$

so this is called the  $n$ th partial sum the  $n$ th partial sum  $ok$

so if we think carefully we will understand that  $s_1$  is equal to  $a_1$   $s_2$  is equal to  $a_1$  plus  $a_2$   $s_3$  is equal to  $a_1$  plus  $a_2$  plus  $a_3$   $a_n$  is equal to  $a_1$  plus  $a_2$  plus sorry  $s_n$  is equal to  $a_1$  plus  $a_2$  up to  $a_n$  thus actually we generate a sequence  $s_1$   $s_2$   $s_n$  and if this sequence has a limit then the series  $a_1$  plus  $a_2$  plus  $a_n$  as  $n$  goes to infinity will converge or the series will converge if the corresponding sequence of partial sums  $s_1$   $s_2$

$s_n$  has a limit for example let  $a_k$  is equal to  $k$  then  $s_n$  is equal to  $1 + 2 + \dots + n$  which is  $n(n+1)/2$  right therefore as  $n$  goes to infinity the  $n$ th partial sum  $s_n$  also goes to infinity therefore the series  $\sum_{k=1}^{\infty} k$  does not converge similarly consider  $a_k$  is equal to  $(-1)^k$  therefore  $a_1 = -1, a_2 = 1, a_3 = -1, a_4 = 1, \dots$  we have already seen  $a_3 = -1, a_4 = 1, \dots$  implies the series does not converge therefore when discussing a series we have to find out whether the series is convergent or not if it does not converge then we cannot get the value of the series and therefore if we want to use an polynomial of arbitrary degree to approximate a function  $f(x)$  we won't be able to approximate that as the series is going to be divergent now let us consider geometric progression all of you are ever that geometric progressions with first term  $a$  and the common ratio  $r$  will look like this right that is the  $k$ th term is equal to  $a r^{k-1}$  we all know the partial sum say  $s_k$

so what is  $s_k$   $s_k$  is equal to  $a + ar + ar^2 + \dots + ar^{k-1}$  therefore  $r s_k$  is equal to  $ar + ar^2 + \dots + ar^k$  that is  $s_k - r s_k$  is equal to  $a - ar^k$  then as I am subtracting they are all getting cancelled out and what is left with is  $a - ar^k$  therefore  $(1-r) s_k$  is equal to  $a(1-r^k)$  therefore  $s_k$  is equal to  $a(1-r^k)/(1-r)$

so we find that for the gp  $s_k$  is equal to  $a(1-r^k)/(1-r)$  upon  $1-r$  or we can write it as  $a(1-r^k) r^{k-1} / (1-r)$  depending upon whether  $r$  is smaller than one or greater than one we get one of the expressions what happens if  $k$  goes to infinity that means we are looking at the geometric progression when the number of terms is infinite consider limit  $k$  goes to infinity  $a(1-r^k) / (1-r)$  if modulus of  $r$  is less than 1 then we know  $r^k$  goes to 0 therefore this term as  $k$  goes to infinity goes to  $a / (1-r)$  as  $k$  goes to infinity on the other hand if modulus of  $r$  is greater than 1 then  $r^k$  goes to infinity therefore limit  $k$  goes to infinity  $a r^{k-1} / (1-r)$  is infinity and therefore the series does not converge right

so this is an important lesson for us if we are talking about  $a + ar + ar^2 + \dots$  or which is equivalent to  $a(1+r+r^2+\dots)$  it will converge if modulus of  $r$  is less than one else it will diverge this brings us to the interesting series  $1 + r + r^2 + \dots$  like that where does it converge if we look at the sum  $a(1-r^k) / (1-r)$  upon  $1-r$  that was the gp series we have got up to  $k$  terms if modulus of  $r$  less than one this will sum up to  $a / (1-r)$  in particular keeping  $a$  is equal to one we can say  $1 + r + r^2 + r^3 + \dots$  this series converges to  $1 / (1-r)$  for example what is  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$  this sum we know that there are infinitely many terms in the series therefore as such we cannot compute the sum but using this result we can say that this sum is  $1 / (1 - \frac{1}{2})$  which is equal to 2 ok

so we can compute this series using the concept of limit that this series value is going to be two in a similar way what is  $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$  that is I am looking at  $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$  that is I am looking at the series  $\sum_{k=0}^{\infty} \frac{1}{3^k}$  which is equal to  $\frac{1}{1 - \frac{1}{3}}$  which is equal to  $\frac{3}{2}$ .

so this series will converge to the value three by two right  $1 - r$  to the power minus 1 is equal to  $1 + r + r^2 + \dots$  for modulus of  $r$  less than one ok now the question is what about  $1 + r$  whole to the power minus 1 we can compute it easily from this

so we are looking at  $1 + r$  whole to the power minus one we can write it as  $1 - (-r)$  whole to the power minus 1 therefore putting the value of  $r$  is equal to  $-r$  in the series expansion we get  $1 + r$  whole to the power minus 1 is equal to  $1 - (-r) + (-r)^2 + (-r)^3 + \dots$  like that this is equal to  $1 - r + r^2 - r^3 + r^4 - \dots$  to the power 4 like that it will go thus if the value of modulus of  $r$  is less than 1 then in a similar way we can find  $1 + r$  whole to the power minus one as well have you seen similar expressions before yes you did binomial expansion if i asked you what is  $1 + r$  whole to the power  $n$  you know the answer is  $1 + nC_1 r + nC_2 r^2 + \dots + nC_n r^n$  where  $nC_k$  is equal to  $\frac{n!}{k!(n-k)!}$  here  $n$  is a same positive integer right

so we know how to expand  $1 + r$  whole to the power  $n$  for a given positive integer  $n$  we also know that there are  $n + 1$  terms in the series since this is finite we can compute uh today i have seen  $1 + r$  whole to the power minus one

so note that the power here is negative therefore what has happened we have seen for positive  $n$  finite sum or finite series but for negative  $n$  we get an infinite series in particular we found that the series  $1 - r$  whole to the power minus 1 or  $1 + r$  whole to the power minus 1 we could compute their values if modulus of  $r$  is less than one now i will stop here but giving you some task try to do these what is  $1 - r$  whole to the power minus 2 or  $1 - r$  whole to the power minus 3.

right okay students in the next class i will start with these problems and i will show how we can obtain the values of such terms for a given value of  $r$  whose modulus is less than one till then you try to practice this we will see you in the next class thank you

so much you