

sequence and series in this lecture we shall explore more on arithmetic mean and geometric mean of the numbers further we will try to tackle certain problems on arithmetic progression and geometric progression recall that given two numbers a and b arithmetic mean a_m for short of a and b is defined by given two positive numbers a and b the geometric mean of a and b is defined as follows positive square root of the product a and b let us see a few instances arithmetic mean of the numbers 1 and 2 is $1 + 2$ divided by 2 which is 1.5 and geometric mean of 1 and 2 is square root of 2.

arithmetic mean of 1 and 4 is $1 + 4$ by 2 which is 2.5 and geometric mean of the numbers 1 and 4 is positive square root of 4 which is 2.

arithmetic mean of 2 and 8 is $2 + 8$ by 2 which is 5 and geometric mean between two and eight is four you can play with more numbers from these instances can you compare the value a_m and g_m of two numbers in the sense which one is larger you can observe that at least in this instance arithmetic mean is greater than or equal to the geometric mean in these cases strictly greater can we dream this inequality in general case that is is it true that arithmetic mean between two positive real numbers is always greater than or equal to geometric mean we shall settle this question next let a and b be two positive real numbers then a_m the arithmetic mean is $a + b$ by 2 and g_m is root a and b we shall ask the question whether a_m is always greater than or equal to g_m is it the case that a_m greater than or equal to g_m that is we would like to know whether $a + b$ by 2 is greater than or equal to root a and b that's the question it amounts to check if the difference a_m minus g_m is non-negative

so let us consider the difference consider $a + b$ by 2 minus root a and b with a simple manipulation it is $a + b$ minus two times root a and b by two completing the numerator into square you can observe that the numerator is root a minus root b the whole square by 2 thus the difference a_m minus g_m is equal to root a minus root b the whole square divided by 2.

note that root a and root b are real numbers their difference is a real number and square of a real number is always non-negative therefore root a minus root b the whole square is non-negative which says that the difference a_m minus g_m is non-negative therefore a_m is greater than or equal to g_m $a + b$ by 2 is greater than or equal to root a and b thus we established a general inequality a_m g_m inequality which says that arithmetic mean of two positive numbers is always greater than or equal to geometric mean between them one may ask the question when equality holds let us go back to our investigation a_m equal to g_m then the difference is zero

so the question reduces to when root a minus root b the whole square coincides with zero and answer is that's when root a equal to root b since we deal with positive numbers it's equivalent to say a equal to b thus the observation is that equality holds if and only if a equal to b to conclude arithmetic mean between two numbers is always greater than or equal to geometric mean between them further the arithmetic mean and geometric mean coincides only when the two numbers are equal let us try to interpret this arithmetic geometric mean relation geometrically consider a rectangle with side length a and b then perimeter of the rectangle will be sum of all sides which is two a plus two b and area is a and b now let us consider a square with area equal to area of this rectangle means we would like to have a square with area a and b note that for a square formula for area is square of the side therefore to have a square with area a and b we need square of all side length root a and b let us consider square of side lengths root a and b then area of this square obviously is a and b and perimeter is four times root a and b sum of lengths of all sides keep this in mind let us apply a_m g_m inequality to the numbers representing the side length a and b a_m g_m

inequality says that $a + b$ by 2 is greater than or equal to \sqrt{ab} which is equivalent to say 2 times $a + 2$ times b is greater than or equal to 4 times \sqrt{ab} by multiplying with 4 on both side thus interpreting am gm inequality in terms of perimeter we observe that among all rectangles with equal area the perimeter of the square is the least compared to perimeter of any other rectangle having the same area recall the right hand side of this inequality is perimeter of the square and left hand side of this inequality represent perimeter of a rectangle

so am gm inequality immediately translates to a geometrical fact namely among all rectangles with equal area the square has least perimeter next let me make a remark arithmetic mean and geometric mean we have defined for two positive real numbers we can generalize this and define arithmetic mean and geometric mean for finite number of real numbers to be precise given n real numbers a_1, a_2, \dots, a_n the arithmetic mean of these numbers is defined as $\frac{a_1 + a_2 + \dots + a_n}{n}$ we add all the numbers and divide by number of real numbers we have similarly given n positive reals a_1, a_2, \dots, a_n

so on a_1, a_2, \dots, a_n the geometric mean of these numbers is defined as follows $\sqrt[n]{a_1 a_2 \dots a_n}$ is equal to the product $a_1 a_2 a_3 \dots a_n$ power $\frac{1}{n}$ of the product of the numbers observe that when n is equal to 2 it reduces to the formula that we had for geometric mean between two numbers let me mention without proof that the am gm inequality holds for a set of n positive reals that is given in positive reals a_1, a_2, \dots, a_n

so on a_1, a_2, \dots, a_n the arithmetic mean of these n numbers is always greater than or equal to the geometric mean it will be a nice exercise to establish this inequality note that we have this inequality in the case of two real numbers positive real numbers which emerged out of a very trivial fact that square of any real number is greater than or equal to zero using the am gm inequality for two real numbers as a basis and applying induction one may try to establish it for n positive reals next let us go back to an infinite series as i told in the previous lectures unlike a finite sum an infinite sum or an infinite series cannot be dealt in a straight forward manner what we do is we find sequence of partial sums next we observe what happens to the sequence of partial sum as n becomes larger and larger if the sequence of partial sum comes close to a fixed real number as n becomes larger and larger we say the series is summable or convergent and that fixed number to which sequence of partial sum comes close will be treated as sum of the series in that context we observe that a geometric series infinite geometric series is summable if the common ratio lies between minus 1 and 1 both excluded further in the case of convergent the sum has a formula given by $\frac{a}{1-r}$ in the other cases namely modulus of common ratio greater than or equal to 1 the geometric series diverges one may ask similar to a infinite geometric series why don't we consider an infinite arithmetic series that is given an arithmetic progression $a, a+d, a+2d, \dots, a+(n-1)d$

so on $a, a+d, a+2d, \dots, a+(n-1)d$ can we talk about the sum $\sum_{n=1}^{\infty} (a+(n-1)d)$ can we talk about sum of all terms of an arithmetic progression whether this series is summable to answer this first let us have the following observation let summation $\sum_{n=1}^{\infty} a^n$ an infinite series $a + a^2 + a^3 + \dots$

so on be convergent meaning roughly you can add all these terms and end up with a finite value in such case you know that the corresponding sequence of partial sum that is sequence of partial sum namely $s_n = a + a^2 + \dots + a^n$ is convergent that is as n becomes larger and larger the sequence of partial sum becomes sufficiently close to a fixed number let us call it yes thus with

the assumption that given series is summable or convergent we have the corresponding sequence of partial sum is convergence this should give you that as n becomes larger and larger both s_n and s_{n-1} are close to s remember when n is large there is no big difference between $n-1$ and n and what we mean by convergence of a sequence is that as we progress towards the end of the sequence all the terms get stagnated near a fixed number yes therefore once s_n is convergent to s both s_n and s_{n-1} will be very close to this fixed s when n is large note that s_n is sum of first n terms $a_1 + a_2 + \dots + a_n$ therefore $a_n = s_n - s_{n-1}$ n th term is same as difference between n th term and $n-1$ term of the sequence of partial sum note that both s_n and s_{n-1} are close to s when n becomes large hence intuitively it should be clear that $\lim_{n \rightarrow \infty} a_n = 0$ as both s_n and s_{n-1} are close to s the difference will be close to 0 when n becomes larger let us not enter into the precise definition of limit of sequence and

so on but have an intuitive feeling that when s_n and s_{n-1} are close to s the difference will be close to zero therefore $\lim_{n \rightarrow \infty} a_n = 0$

so what do we have we started with the fact that the infinite series is convergent that infinite series finally represent a finite real number $a_1 + a_2 + a_3 + \dots$ amounts to some s in that case we conclude that the terms a_n should become close to 0 thus summation a_n is convergent implies $a_n \rightarrow 0$ as $n \rightarrow \infty$ recall that if you have a statement p implies q it is logically equivalent to $\neg q$ implies $\neg p$ be careful p implies q is logically equivalent to $\neg q$ implies $\neg p$ going back to our observation series summation a_n is convergent implies the terms become close to 0 as n becomes larger and larger hence applying this logical equivalence one should be able to observe that if n th term does not tend to zero as n becomes larger and larger we cannot expect that the corresponding series is convergent if a_n does not come close to 0 as $n \rightarrow \infty$ then summation a_n is not convergent in other words that summation cannot represent a finite value it is not summable this is going to be one of the powerful tests to see whether a series is divergent meaning not convergent if you observe that terms in the series are not becoming close to 0 as n becomes larger and larger immediately you can conclude that the corresponding series is not summable keeping this observation let us go back to the question given an arithmetic progression $a, a+d, a+2d, \dots$ does the infinite series summation $\sum_{n=1}^{\infty} (a + (n-1)d)$ converge does this infinite series finally represent a number real value note that for this series n th term is $a + (n-1)d$ bear in mind that a and d are fixed finite real numbers which are respectively first term and common difference of the AP once a and d is fixed assuming $d \neq 0$ we see that $a + (n-1)d$ becomes large when n becomes large large in magnitude if d is a positive number then $(n-1)d$ as n becomes large comes close to infinity and if d is negative $(n-1)d$ when n becomes large comes close to minus infinity therefore $\lim_{n \rightarrow \infty} (a + (n-1)d) = +\infty$ or $-\infty$ what we observed is that n th term of the series that we are interested namely arithmetic series does not come close to 0 when n becomes large therefore by the previous observation we had the corresponding series is not summable therefore summation $\sum_{n=1}^{\infty} (a + (n-1)d)$ is not convergent as the n th term of the series does not come close to 0 when n becomes large the corresponding series is not convergent this is the case if $d \neq 0$ and if $d = 0$ again when $d = 0$ n th term of the infinite series we are interested reduces to a which is fixed therefore as $n \rightarrow \infty$ the n th term does not become close to zero when a is not zero $\lim_{n \rightarrow \infty} a$ which is the n th term is not 0

if a is not 0 what we conclude is that if the common difference is 0 and the first term is not 0 then the arithmetic progression is a, a, a and

so on here the n th term is not coming close to zero therefore the corresponding series $a + a + a + \dots$

so on is not finite or it's not convergent only case left with is a equal to 0 and d equal to 0 in that case the arithmetic progression is $0, 0, 0$

so on and the corresponding arithmetic series is $0 + 0 + 0 + \dots$

so on and which obviously is summable and the sum is 0 .

to conclude except for the trivial case the series corresponding to arithmetic progression namely summation $a + n$ minus 1 into d is not convergent this is in contrast to geometric series a geometric series that is a series corresponding to a geometric progression is convergent for some values of r more precisely for r lying between -1 and 1 both excluding the geometric series is convergent let me stress on the observation we had if summation a^n is equal to 1 to infinity is convergent then that implies that a^n is close to 0 as n becomes large written mathematically as $\lim_{n \rightarrow \infty} a^n = 0$ is equal to zero how do we utilize this result is by taking the contrapositive of the statement namely if you have a series summation a^n equal to 1 to infinity a^n and if you could observe that a^n the term and the term is not close to 0 as n becomes large immediately we can conclude that the series is not convergent however if n th term becomes close to 0 as n becomes large that does not guarantee anything about convergence of the infinite series summation a^n equal to 1 to infinity a^n keep in mind that if a^n becomes close to 0 as n grows arbitrarily large then we cannot conclude that summation a^n is convergent very important remark and to supplement this remark let me give you an example let us consider the sequence $1, 1/2, 1/3, \dots$

so on $1/n$ by n

so on corresponding series will be summation $1/n$ namely $1 + 1/2 + 1/3 + \dots$ recall that we proved the partial sum of this series has the property that 2^n is greater than or equal to $1 + n$ by 2^{n-1} partial sum is greater than or equal to $1 + n/2$.

so the sequence of partial sum is not bounded it goes on increasing therefore we cannot expect that the sequence of partial sum remains close to a fixed number that is equivalent to say the series is not convergent in other words $1 + 1/2 + 1/3 + \dots$ does not represent a finite real number however the n th term namely $1/n$ becomes close to 0 as n becomes large enough $\sum 1/n$ is not convergent but $1/n$ comes close to 0 as n tends to infinity thus when you have a series summation a^n and when the n th term a^n does not come close to zero conclusion is immediate the series is not convergent whereas if the n th term goes to 0 we cannot claim anything about the series it will be more clear if i give one more example let me consider $1 + 1/2 + 1/4 + 1/8 + \dots$ that is the sequence $1/2^{n-1}$ and the corresponding series summation n is equal to 1 to infinity $1/2^{n-1}$ it's not hard to see that this is a geometric series with first term as 1 and common ratio for the corresponding geometric progression is $1/2$ which is less than 1 .

therefore by the observation we had earlier this series is convergent and in fact we have a formula for its sum $a/(1-r)$ type now you observe that the n th term namely $1/2^{n-1}$ becomes close to 0 as n becomes large enough

so in this example a^n tends to 0 as n tending to infinity and summation a^n is convergent whereas in previous example a^n tends to 0 as n tending to infinity but summation a^n is not convergent to sum up if n th term of a series is not

converging to 0 immediately conclude the series summation a_n is not convergent and if it does converge to 0 we cannot conclude something about the corresponding series summation a_n don't bother too much about the technical details about the convergence divergence of the series but try to have a intuitive feeling of it having said that let us move to some problems based on the concept we discussed

so far this problem should help you to remind you about the formulae that we developed in ap and gp and it should supplement your theoretical understanding first problem is this sum of n terms of 2 ap are in the ratio $3n + 8$ by $7n + 50$ you are asked to find the ratio of their twelfth term the data is ratio of sum of n terms of 2 ap please recall that given an ap with first term a and common difference d we do have a formula to find sum of n terms of that ap since we have to deal with two ap's here let us assume that first ap arithmetic progression has first term let us call a_1 and common difference d_1 then the ap will be $a_1, a_1 + d_1, a_1 + 2d_1$ and

so on let's second ap has first term a_2 and common difference d_2 then the second ap will look like $a_2, a_2 + d_2, a_2 + 2d_2$ and

so on n th term of first ap is first term plus $n - 1$ into common difference n th term of second ap is first term plus $n - 1$ times common difference d_2 these formula we developed already now what is given to us is ratio of sum of n terms of 2 ap recall that sum of first n terms of ap is given by the formula $\frac{n}{2} [2a + (n - 1)d]$ this is for sum of n terms of first ap similarly sum of n terms of second ap will be $\frac{n}{2} [2a_2 + (n - 1)d_2]$ what is given to you is ratio of these two quantity $\frac{\frac{n}{2} [2a_1 + (n - 1)d_1]}{\frac{n}{2} [2a_2 + (n - 1)d_2]}$ is given to be $\frac{3n + 8}{7n + 50}$ canceling this $\frac{n}{2}$ the given fact is $\frac{2a_1 + (n - 1)d_1}{2a_2 + (n - 1)d_2}$ is equal to $\frac{3n + 8}{7n + 50}$.

remember a_1 and d_1 the first term and common difference of first ap a_2 and d_2 the first term and common difference on second ap are unknown and this will give you after simplification only one equation and there are many unknowns

so we cannot expect it to be solved however let us see what's the question question is about finding ratio of twelfth term remember we have formula for n th term of first ap and n th term of second ap therefore twelfth term of first ap will be $a_1 + 11d_1$ and 12th term of second ap will be $a_2 + 11d_2$ we are asked to find the ratio of this $\frac{a_1 + 11d_1}{a_2 + 11d_2}$ this is what we are asked and what we have is $\frac{2a_1 + (n - 1)d_1}{2a_2 + (n - 1)d_2} = \frac{3n + 8}{7n + 50}$ that's what we have finding the ratio $\frac{a_1 + 11d_1}{a_2 + 11d_2}$ is equivalent to finding $\frac{2a_1 + 22d_1}{2a_2 + 22d_2}$ multiply it numerator and denominator with 2 now see what we have what we have is ratio of $\frac{2a_1 + (n - 1)d_1}{2a_2 + (n - 1)d_2}$ in place of $n - 1$ we need 22 that is n is equal to 23 therefore the required ratio can be obtained by putting n equal to 23 in the given equation putting n equal to 23 in star by star i mean this equation putting n equal to 23 we have $\frac{2a_1 + 22d_1}{2a_2 + 22d_2} = \frac{3 \times 23 + 8}{7 \times 23 + 50}$ now simplifying this ratio one may get the answer i think the answer is $\frac{7}{16}$ please do the manipulation and confirm we shall continue with more problems on ap and gp in the next class thank you you