hi all welcome back to the series of lecture on the topic sequence and series this is our sixth lecture in this topic let us begin by recalling that the two words sequence and series are used interchangeably in day-to-day life both these words are used to indicate succession of event that's why i told series of lectures however it should be born in mind that these two words sequence and series have distinct meaning in mathematics roughly sequence is used to indicate list or succession of numbers and series is used to designate sum of terms of a sequence having said this as promised in the last lecture we shall establish a formula for sum of n terms of a gp recall that a gp geometric progression is a sequence in which each term after the first term is obtained from the previous term by multiplying with a fixed non-zero number and that fixed non-zero number is termed common ratio of the geometric progression next we shall establish a formula for sum of n terms of a geometric progression let us consider a gp with first term a and common ratio r note that a gp with first term a and common ratio r can be explained or represented by the list a a r a r square etc nth term a r power n minus 1 etc our goal is to find a formula for a plus a r plus a r square plus etc up to the nth term namely a into r power n minus 1 let us denote this sum by s n what we intend to do is to find a formula for sn let us first settle the trivial case namely let r equal to 1 note that in this case the geometric progression reduces to the constant sequence a a a

so on consequently sm is a plus a plus

so on plus a n times that is sn is equal to n times a this settles the trivial case now let us consider the remaining case let r not equal to 1 what we want is a formula for s n equal to a plus a r plus etc up to a r power n minus 1 we shall use a simple trick as follows let us find r times s which coincides with a r plus a r square plus etc the last term when multiplied with r becomes a r power n let me write the previous term for you it will be a r power n minus 1.

do you see it now let us subtract the second equation from the first that is s n minus r s n equal to see that the terms are a r square etc a r power n minus 1 cancels in this subtraction process we end with a minus a r power n the left hand side boils down to 1 minus r times sn and the right hand side can be simplified to a times 1 minus r power n from this we can easily isolate s n recall that by assumption r is not 1

so division with 1 minus r is possible doing

so we get sn is equal to a times 1 minus r power n by 1 minus r to sum up sum of first n terms of a geometric progression with first term a and common ratio r is n times a if r is equal to 1 and a into 1 minus r power n by 1 minus r if r is not equal to 1 observe that when r is not equal to 1 we can write the formula also as a into r power n minus 1 by r minus 1 just by multiplying numerator and denominator with minus 1

so this is a nice formula for sum of n terms of a gp i hope this is a good time to remind you about an infinite sum or a series to be more precise what are all the troubles in extending the concept of finite sum to an infinite sub note that when we have a finite sum that is sum of finitely many real numbers we can add first two of them and to this sum we can go on adding one term each time the process will terminate and we will be getting a finite value further when we have a finite sum the order in which the terms are added does not really matter whereas if we have an infinite sum note that we cannot go on adding one term at a time to see what comes out for the obvious reason that the process of adding one term at a time will not terminate secondly unlike the case of a finite sum we observe that in an infinite sum the order in which the terms are considered while performing the addition matters in other words to deal with sum of infinitely many real numbers first the real numbers should be ordered in some definite manner and ordering of real numbers give rise to a sequence thus to

define an infinite sum we should start with a sequence of real numbers rather than just a set of real numbers given a sequence of real numbers a n n is equal to one to infinity the expression a1 plus a2 plus etc which can be written for brevity using summation notation as sum n equal to 1 to infinity a n is called a series now how do we assign a meaning for this expression recall that to assign a definite meaning for this expression we first find the sequence of partial sums sn is equal to a1 plus a2 plus etcetera plus a n thus we have a new sequence sn n is equal to 1 to infinity which originates from the given sequence a now we observe what happens to this sequence of partial sum as n becomes large to be precise we find limit n tending to infinity s whether the terms of the sequence become close to a fixed real number as n becomes larger and larger that's what we investigate if this limit exists and if it is yes then this yes is regarded as sum of that series or precisely the value of the series we write s is equal to summation n is equal to 1 to infinity a n and we say series a n is summable or more technically convergent thus to see whether an infinite sum has a meaning in other words whether it is summable in other words whether a series is convergent we should first find sequence of partial sums then we investigate what happens to this sequence of partial sum as n becomes larger and larger keeping this in mind let us try to tackle some of an infinite series namely a geometric series by a geometric series we mean series originating from geometric progression remember a geometric progression is a sequence of the form a a r a r square and

so on now we deal with its sum series of the form a plus a r plus etcetera plus a r power n minus 1 plus etcetera the infinite series which can be written as summation a r power n minus 1 n is equal to 1 to infinity is called geometric series now the question is whether a geometric series is some mobile whether it is convergence note that some ability of a series amounts to finding whether the sequence of partial sums is convergent or not as far as this geometric series is considered the sequence of partial sum namely the sum of first n terms is already found we have a formula for it note that s n the sequence of partial sum is n a if r equal to 1 and a into r power n minus 1 by r minus 1 if r is not equal to 1 hence to answer whether a geometric series is summable that is whether finally it represents a finite number or not it is enough to investigate what happens to this sequence of partial sum s n when n becomes larger and larger let us do that let r is equal to 1 the common ratio is 1 note that in that case s n is equal to n a recall a is a fixed real number therefore as n becomes larger and larger sn namely n a becomes large in magnitude it should be clear that as n tending to infinity n a tends to infinity or becomes very large or becomes very small depending upon sine of a let me write limit n tending to infinity s n is equal to plus or minus infinity depending on sine of a this is the case r equal to 1

so in this case we observe that the sequence of partial sum is not becoming close to a fixed real number finite real number in technical language limit sn is does not exist therefore the geometric series is not convergent or not summable in this case let us take the special case r equal to minus 1 next in this case the geometric progression becomes a next is a r which is minus a next is a r square which is a and

so on consequently s 1 will be a sequence of partial sum first one second sequence of partial sum will be first term plus second term a plus minus a which is 0 the third sequence of partial sum will be a plus minus a plus a which is a and

so on you should be able to observe that the sequence of partial sum s n alternate between a and 0 what i mean is it takes these two value alternatively intuitively from this it should be clear that as n becomes larger and larger s n

does not become close to any number it will go on oscillating between a and 0 it will be either a or 0 it will not remain fixed close to a fixed number

so in this case limit n tending to infinity s n does not exist as the sequence of partial sum does not have a limit we conclude that in this case the geometric series summation a r power n minus 1 is not summable roughly this series does not represent a finite number in case r is equal to minus 1.

now let us take the other case r is neither one nor minus 1 observe that in this case sequence of partial sum s n takes the formula a into 1 minus r power n by 1 minus r what we have to observe is what happens to this sn when n becomes larger and larger note that s n can be written as a by 1 minus r minus a r power n by 1 minus r obviously the first term is independent of n

so to investigate what happens to s n as n becomes large enough it is enough to investigate what happens to the second term as n becomes larger to be more precise it is enough to see what happens to r power n as n becomes larger and larger to do

so let us split the values of r which we are interested namely r not equal to 1 comma minus 1 into two categories one mode are strictly less than 1 that is we are interested in those values of r lying between -1 and 1 both excluded when mod r is less than 1 of course r can take both positive and negative value but then since it is less than 1 in magnitude it will be of the form 1 by some m r will be of the form 1 by m where m is fixed do you agree

so as n becomes larger and larger r power n that is 1 by m power n becomes smaller and smaller do you see it because in 1 by m power n the denominator becomes large enough as n becomes larger

so my conclusion is that in this case limit n tending to infinity our power n is zero let me repeat we are interested in r values which is fixed but lying between minus 1 and 1.

in that case r can be thought of as some number of the form 1 by m m may be positive or negative doesn't matter

so r power n will be something of the form 1 by m power n m is fixed now when n becomes larger the denominator becomes very large

so that 1 by m power n become close to 0 that is how intuitively you can argue limit n tending to infinity r power n equal to zero due to this going back to s n we can observe that as n becomes arbitrarily large the second term become close to 0 as we have limit r power n as n tending to infinity is zero therefore the second term contributes nothing as n becomes large and we deduce that limit n tending to infinity s n boils down to a by 1 minus r if you recall the definition of an infinite series or more specifically convergence of an infinite series limit of partial sum is what we call as sum of the series therefore in this case we conclude that summation a r power n minus 1 n is equal to 1 to infinity sum of all terms of a g p is a by 1 minus r

so far we discuss only the case r equal to minus 1 r equal to 1 and the case r lies between minus 1 and 1.

now let us take r to be some fixed number outside this values that is let mode are greater than one r can be positive or negative but in magnitude it is greater than one now see that limit n tending to infinity mode r power n since r is outside minus 1 1 mod r is greater than 1

so as n tending to infinity you have that mod r power n becomes large and large something like 2 power n you can see that 2 4 then 8 then 16 and

so on as n increases the power increases indefinitely therefore this is infinity keeping this in mind and going back to the expression of sn one can see that as n becomes large this term namely the second term a r power n by 1 minus r does not become close to a fixed real number in other words it is not convergent therefore limit s n also does not exist therefore limit n tending to

infinity s n does not exist to put in language of series this is equivalent to say that the corresponding series is not summable that is summation a r power n minus 1 n is equal to 1 to infinity is not convergent in the case mod are greater than one let us sum up for a geometric progression we have an expression for sum of first n terms and the expression is sn is equal to a into 1 minus r power n by 1 minus r for r not equal to 1 and for r equal to 1 it reduces to a trivial case namely sn is equal to n times a now considering an infinite geometric series summation a r power n minus 1 in ranging from 1 to infinity we see that when r equal to 1 the series is not convergent when r equal to minus 1 the series is not convergent the geometric series do represent a finite value when r lies between minus 1 and 1 and that finite value is a by 1 minus r and for mode r greater than one the geometric series does not represent a finite value or more technically the geometric series is not convergent let me record this observation here a geometric series of the form summation n equal to 1 to infinity a r power n minus 1 is convergent if mod r is less than 1 further for mod r less than 1 mod r less than 1 the sum of all terms is a by 1 minus r for other values of r that is mod r greater than or equal to 1 summation n equal to 1 to infinity a r power n minus 1 is not convergent i would strongly recommend you to re remember this result an infinite geometric series is convergent if the common ratio lies between minus 1 and 1 excluding these two cases and in that case the geometric series sum to a by 1 minus r and for mod r greater than or equal to 1 the geometric series is not summable let me make a passive remark here note that in the case of an infinite geometric series we observed when it converge and when it does not moreover in case it is converge what is sum of a geometric series these things we observed unlike geometric series in many other infinite series the question of whether the series is summable or convergent even if it is convergent what is its sum these two questions are tackled in two steps in case of many infinite series first we shall investigate whether the series is convergent or not in case it's found to be convergent most of the cases we have to be satisfied with some estimate for its sum rather than getting an expression for some in other words a formula for sum of infinite series like that in the case of geometric series is rare let us proceed recall that given two real numbers a and b we asked a question can we insert a number capital a between a and b

so that these three numbers forms terms of an arithmetic progression in fact we had a formula for a a plus b by 2 and we called this number as arithmetic mean of a and b am for short we shall ask a similar question but now instead of ap hgp that is the question reads as given two real numbers a and b does there exist a number let us call g such that a g and b forms a gp hope you understand the question you are supplied with two numbers for the time being let us not impose any condition on a and b except that they are reals question is can we always find a number g such that a g and b forms terms of a geometric progression let us settle this question note that if a g and b are terms of a gp consecutive terms of a gp the ratio of second term by faster should coincide with the ratio of third term by second term that is g by a coincide with b by g if a g b are in geometric progression that is equivalent to say g square is equal to a b

so we are asking the question does there exist a number g such that g square is equal to a b note that square of a real number is always non-negative therefore for existence of a real g satisfying g square is equal to a b a and b should have same sign

so let us slightly modify the question and ask ourselves given two positive numbers a and b does there exist a g such that a g b forms a gp answer is yes take g to be root of a b in this case a g b forms a geometric progression and

this g is referred to as geometric mean of given positive numbers let me record it as a definition given two positive numbers capital a let me write small a and small p small p the geometric mean gm for short of a and b is defined as gm of a and b is equal to root of a b this is similar to arithmetic mean in case of arithmetic mean we have addition and division we add and divide by 2 here you have multiplication and taking exponents taking powers to sum up given two positive numbers a and b you can always get a number capital g namely the geometric mean of these two numbers

 so that a capital g b forms a geometric progression now let us generalize this a bit and ask the following question given two positive numbers a and b can we insert as many numbers as we need but finite such that a g1 g2 etc gn b forms a geometric progression given two positive real numbers a and b we would like to insert n real numbers which we designate g1 g2 etc gn

 so that a g1 g2 etcetera g and b form say gp towards the answer let us observe the following if we need a g1 g2 etcetera g and b to form a geometric progression more precisely consecutive terms of a geometric progression b should be n plus 2 term of that geometric progression b is n plus two at the term we have a formula for nth term of a gp with first term a and common ratio are letting common ratio of the desired gp to be r the n plus 2 term is given by the formula a r power n plus 2 minus 1 that we need to be b

 so r power n plus 1 is equal to b by a which amounts to r equal to b by a the whole power 1 by n plus 1.

 now we have the first term a and common ratio this thing b by a power 1 by n plus 1 once we have first term and common ratio we can completely specify what is the geometric progression is therefore g 1 being the second term of the geometric progression will be a times the common ratio which is a times b by a the whole power 1 by n plus 1.

 similarly g 2 will be being third term of the geometric progression it will be a times r square which is a times b by a the whole power 1 by n plus 1 square and

 so on thus we conclude that given two positive numbers it is always possible to insert finitely many real numbers between them

 so that the list forms a gp we shall continue with gp and ap in the next lecture thank you you