

welcome back to the lecture on sequence and series to begin with let us recall an arithmetic progression which we defined in the last lecture let us call a_n in short recall that an arithmetic progression is a sequence such that the difference of any two consecutive terms remain the same to put in simple a sequence a_n n ranging from 1 to infinity is called an arithmetic progression if a_{n+1} is equal to $a_n + d$ for all integers greater than or equal to 1.

this essentially says that the difference of two consecutive terms namely a_{n+1} and a_n is d where d is a constant the first term a_1 and the common difference d determines the arithmetic progression completely that is if a_n is equal to 1 to infinity is an arithmetic progression with a_1 is equal to a called first term and the difference between two successive terms is d called the common difference then arithmetic progression can be written in standard form as follows the second term will be $a + d$ the third term will be $a + 2d$ and

so on observe that between n terms there are $n - 1$ successive common differences hence or otherwise following this pattern it is not hard to see that in term of this a_n namely a_n with first term a and common difference d is given by the formula $a + (n - 1)d$ next let us observe some properties of an arithmetic progression of which we have discussed first property in the last lecture let me remind you if a_n is equal to 1 to infinity is an arithmetic progression then sequence b_n obtained by adding same number to each term of the given a_n that is the sequence b_n where n th term b_n is equal to $a_n + k$ for every n is again in arithmetic progression is arithmetic progression essentially this property says that we can create an arithmetic progression from a given arithmetic progression by adding a constant to each term second property is very similar instead of adding a constant to each term we can subtract or in other words this k can be positive or negative to be explicit let me list it if a_n is equal to 1 to infinity is an arithmetic progression then the sequence b_n obtained by subtracting a constant say k to each term is again a arithmetic progression

so let us write b_n is equal to $a_n - k$ for every n starting from 1 the claim is that sequence b_n is an arithmetic progression to continue let us have a similar property if sequence a_n is equal to one to infinity is an arithmetic progression or an arithmetic sequence then sequence obtained by multiplying each term of sequence a_n with a constant is again an arithmetic progression though it is straight forward let us work out the details our assumption is that a_n is an a_n it means $a_{n+1} - a_n$ is a constant let us call it as d for every n element of set of natural number n using this given sequence let us construct a new sequence b_n consider sequence b_n n equal to 1 to infinity how do we construct b_n we just multiply a_n with a constant let b_n is equal to some c times a_n for every n we have to observe that this sequence b_n is again in arithmetic progression let us do that consider the difference of two successive terms in b_n namely $b_{n+1} - b_n$ this will be $c(a_{n+1} - a_n)$ which is c times d as a_n forms an arithmetic sequence $a_{n+1} - a_n$ is a constant for all n hence we get c times d

so what we observe for all natural numbers n the difference $b_{n+1} - b_n$ remains a constant this establishes the fact that sequence b_n is an arithmetic sequence but unlike in the other case were we constructed a sequence from the given arithmetic progression by adding constant to it here the common difference of the constructed sequence is different from the common difference of the given arithmetic progression see that the difference $a_{n+1} - a_n$ is d whereas the difference $b_{n+1} - b_n$ is not the same d but it may be different from d depending upon what is this c similar result can be told for division let

me quickly write it here if sequence a_n is equal to 1 to infinity is an arithmetic progression then sequence b_n obtained by dividing each term of sequence a_n with a non zero constants remains to be an ap given an ap you can divide each term of that ap with a non-zero number you can obtain a new sequence and it is not hard to see that the new sequence is again an arithmetic progression to sum up we can construct an ap from a given ap by adding constant to each term by subtracting a constant from each term by multiplying a constant with each term or by dividing each term by a constant in the case of division make sure that the number with which you are dividing is non-zero now let us ask the following question let a, b be given numbers given real numbers can we insert a number let us call it as capital A such that a, A, b are terms of an arithmetic progression

so this is the question we would like to address right now you are given with two real numbers let us denote it by small a and small b and we are asked to see whether we can come up with a capital A such that a, A, b becomes three consecutive terms of an arithmetic sequence observe that for an arithmetic progression or an arithmetic sequence the difference of any two successive term should remain the same with that basic principle or we can answer this question since a, A, b are to be in ap the difference namely $A - a$ should coincide with the difference $b - A$ this reads off $2A = a + b$ which provides $A = \frac{a + b}{2}$ thus given two real numbers a and b it's always possible to get a number capital A such that a, A, b are terms of an arithmetic progression and the capital A is given by $\frac{a + b}{2}$.

let us make a definition here given two numbers a and b the number $\frac{a + b}{2}$ is called arithmetic mean A_m for short of a and b let me put in simple arithmetic mean of a and b is given by $\frac{a + b}{2}$ you have observed previously that when two numbers are given and we insert arithmetic mean between them then the three numbers can become three consecutive terms of an arithmetic progression having said this let us ask a slightly general question question is this instead of inserting one number between given two numbers a and b can we insert finite number of real numbers

so that the given two numbers and the inserted numbers all together can successive terms of some arithmetic progression let me write it for you let a and b be two numbers given what is the question can we insert numbers not just one number but finite number of real numbers let us call a_1, a_2, \dots, a_n

so that $a, a_1, a_2, \dots, a_n, b$ are successive terms of a sequence which is and ap

so given two numbers a and b we would like to insert n numbers between them

so that the $n + 2$ numbers all together are successive terms of an arithmetic progression let us try to answer this recall that the requirement is $a, a_1, a_2, \dots, a_n, b$ are in arithmetic progression means these terms are some consecutive terms of an arithmetic progression

so all together we have $n + 2$ terms with b as the $n + 2$ th term of that ap recall given the first term a and common difference d $n + 2$ th term can be obtained using a formula the $n + 2$ th term by formula is $a + (n + 2 - 1)d$ where d is common difference of the ap for which these $n + 2$ number should become consecutive terms that is b should be equal to $a + (n + 2 - 1)d$ since b turns out to be the $n + 2$ th term b should coincide with $a + (n + 2 - 1)d$ working out this gives $d = \frac{b - a}{n + 1}$ recall to describe an arithmetic progression completely what we need is the first term and the common difference here the first term is given number a and we have just now obtained the common difference d as $\frac{b - a}{n + 1}$

thus the second term that is the first number to be inserted capital a_1 will be $a + d$ which is $a + b - a + d$ the second number to be inserted namely a_2 that is going to be the third term in this arithmetic progression hence a_2 is $a + 2d$ which is $a + 2(b - a) + d$ in a similar fashion you can write a_3 which is $a + 3d$ and that will be $a + 3(b - a) + d$ plug in the formula for d namely $b - a + d$ and

so on the last number that we wish to insert in simple a_n is equal to $a + n(b - a) + d$ which is $a + n(b - a) + d$ thus given any two numbers we can always insert finitely many real numbers between them

so that the given numbers along with this inserted numbers can turn out to be successive terms of an arithmetic progression

so let me sum up we have looked into the definition of an arithmetic progression a sequence such that difference between any two successive terms remains constant is said to be an arithmetic progression standard form of an arithmetic progression with first term small a and the common difference d is $a + d, a + 2d$ and

so on the n th term a_n is given by the formula $a + n(b - a) + d$ given two numbers a and b arithmetic mean of a and b is given by the formula $\frac{a + b}{2}$ this is just for a quick recap next let us ask the following question let an arithmetic progression with first term a and common difference d is given let us take it in its standard form $a, a + d, a + 2d$ and

so on the question that we want to answer is the following what is the sum of first n terms of this AP in simple we want to get an expression for $a + a + d + a + 2d + \dots + a + n(b - a) + d$ can we have a closed form expression for this that's what we want to investigate next before answering this let me share an anecdote with you this is a famous story about Carl Friedrich Gauss popularly known as prince of mathematics the story goes like this Gauss was punished by his teacher for his misbehavior what was the punishment the punishment given was find the sum of first hundred natural numbers

so easy to do it right but let me remind you this was when Gauss was fairly five years old surprisingly Gauss could come up with an answer in matter of seconds now how on earth he could come up with some of first hundred natural number within second that too at the age of five years he used a brilliant trick and the trick is grouping these hundred numbers let me give you an outline he listed the hundred natural numbers one two three etcetera up to hundred may not be written fully but then he could see that these 100 numbers can be grouped in the following way group 1 and 100 together group 2 and 99 together group 3 and 98 together and

so on is pairing the hundred numbers do you observe that sum in each pair is one not one the sum of 1 and 100 is one not one sum of 2 and 99 is one not one sum of 3 and 98 is one not one and

so on

so some in each pair is one not one how many pairs are there recall there are hundred numbers

so there are fifty pairs and each pair sums to one not one

so total sum will be 50 pairs are there with each pair summing to one not one

so it is 50 into one not one which is $50 \times 1 = 50$ this is how Gauss could compute sum of first hundred natural number in matter of second no doubt it is a remarkable fact that at the age of five years he could conceive the idea and the same idea of brilliantly pairing and adding is what we are going to do to answer the question I posed slightly before what was the question what is sum of n terms of the arithmetic progression $a, a + d, a + 2d$ and

so on let us answer this with the idea Gauss used to sum one two hundred let us

denote the sum of first n terms of an arithmetic progression by s_n

so s_n is notation I use for $a + a + d + \text{etcetera} + \text{the } n\text{th term}$ remember $n\text{th term}$ of an AP with first term as a and common difference as d is $a + n - 1 \text{ into } d$ and what we are looking for is a formula for this sum $s_n = a + a + d + \text{etcetera} + a + n - 1 \text{ into } d$ since we have only n terms in our consideration let me call the $n\text{th term}$ $a + n - 1 \text{ into } d$ as the last term and let me denote it by l

so we want to find $s_n = a + a + d + \text{etcetera} + l$ remember the idea of summing first hundred natural number we have grouped it adapting it may be with a slight modification we can write s_n also as $a + a + d + \text{etc} + l$ this can be written also as the last term l plus the term previous to l can you tell what it will be $l - d$ and isn't it plus previous to it $l - 2d$ and

so on we will reach up to the first term a

so instead of writing from a to l we are writing from l to a now let us add both this expression s_n sum of first n terms starting from a and ending with l and sum of same n terms but now starting from l and ending with a let us sum these two left hand side gives you $2s_n$

so $2s_n = a + l + a + l + \dots$ do you see that a and l adds up to $a + l$ similarly the second term $a + d$ in the first expression and the second term $l - d$ in the second expression adds up to give $a + l$ it gets cancelled and

so on the last term in the first expression is l and the last term in the second expression is a which adds up to $a + l$ please look at $1 + 2$ and $2 + 1$ again and compare the corresponding terms in the right hand side expression first term in first expression is a first term in second expression is l they added up to $a + l$ second term in right hand side of first expression is $a + d$ and second term in right hand side of second expression is $l - d$ they add up to give $a + l$ and

so on

so $a + l + a + l + \dots$ and

so on $a + l$ and how many are there n terms thus we get $2s_n = n(a + l)$ is equal to n times $a + l$ we want an expression for s_n rather than $2s_n$

so let us write $s_n = \frac{n}{2}(a + l)$ and this gives you a formula sum of first n terms of an arithmetic progression is given by $\frac{n}{2}(\text{first term} + \text{last term})$ in consideration it is an important formula it can also be written as $s_n = \frac{n}{2}(a + \text{remember } l \text{ is the last term in consideration we are considering } n \text{ numbers})$

so l is actually the $n\text{th term}$ of that AP which is $a + n - 1 \text{ into } d$

so by a little arithmetic we can see that it is $\frac{n}{2}(2a + n - 1 \text{ into } d)$ this gives an alternative formula for a sum of first n terms of an arithmetic progression first one can be used if you are supplied with an AP and precisely we can detect the first term and last term whereas the second one we can use if first term and common difference is known and number of terms is given having said these on AP let us proceed to another type of sequence special sequence called geometric progression recall that in arithmetic progression the difference of two terms to successive terms remains a constant instead of difference of any two successive terms if the ratio of any two successive terms of a sequence remains constant we call that sequence as geometric progression let me write the definition precisely a sequence a_n n is equal to 1 to infinity is said to be a geometric progression $g.p.$ for short if no terms are zero and $a_n \neq 0$ plus 1 by a_n the ratio of two successive terms is equal to r for every n element of n let me repeat a sequence a_n of non zero real numbers is said to be a geometric progression if the ratio $\frac{a_{n+1}}{a_n}$ remains constant irrespective of n you take ratio between any two successive term a_n divided

by its preceding term it should remind the constants please note the condition that no term of the sequence is 0 that facilitates this division a_{n+1} by a_n for instant consider the sequence 3 6 12 24 etc can you observe the pattern the second term is first term multiplied with two third term is second time multiplied with 2 and

so on to put in other words the second term by first term 6 by 3 is same as the third term by second term 12 by 6 is same as fourth term by third term and

so on here a_{n+1} by a_n is equal to 2 for every n with the assumption that this pattern follows similarly consider the sequence one by two one by four one by eight one by sixteen etcetera let me write a general term one by two power n etcetera 1×2^n etcetera here also you can see that ratio of two successive terms remains the constant let us recall the definition a sequence of non-zero terms is said to be a geometric progression if a_{n+1} by a_n is equal to r for every n element of n this r the ratio of any two successive terms which remain constant is called common ratio similar to an arithmetic progression were the first term and the common difference completely describes the progression in case of gp the first term and common ratio completely describes a geometric progression if first term is a and common ratio is r then we can write gp in standard form as $a r^{n-1}$ remember a_{n+1} by a_n is r

so a_{n+1} is r times a_n

so the second term will be r times the first term the third term will be r times the second term which is $r^2 a$ and

so on

so standard form of a geometric progression with first term a and common ratio r is given by $a r^{n-1}$ and

so on following the pattern you can see that the n th term of this gp in standard form is the following $a r^{n-1}$ this is the expression for n th term of a gp with the common ratio r and first term a similar to the case of arithmetic progression let us ask the following question can we get a closed form expression for sum of first n terms of a gp consider the gp $a a r a r^2$ etc n th term $a r^{n-1}$ etc can we get a formula for S_n denotes the sum of first n terms $a + a r +$ etcetera $+ a r^{n-1}$ we will develop the formula in the next class this here we will adopt a slightly different technique recall that in the case of ap we use the technique or the trick of grouping properly here we may use a different technique to develop a formula for S_n next class we shall establish the formula and explore gp and ap further thank you