

i welcome you all to this fourth lecture on the title sequence and series to begin with let us recall a definition we made at the end of previous lecture namely the definition of a series given a sequence a_n consider the expression $a_1 + a_2 + a_3 + \dots$ this expression is what we mean by a series associated with the sequence a_n by this time the distinction between sequence and series should be clear sequence is an ordered list of numbers and series is a sum let me remark also that while considering the expression namely sum of infinitely many numbers $a_1 + a_2 + a_3 + \dots$ we are not bothered whether this expression finally represents a real value or not those questions will be answered or discussed in other notions connected with series by a series we just mean this expression or a formal sum $a_1 + a_2 + a_3 + \dots$ also let us note that by definition a sequence is a function from a subset of non-negative integers if that subset is finite we get a finite sequence

so that in this collection of terms a_1, a_2, a_3, \dots there will be only a finite number of terms if we deal with a function from a subset of set of non-negative integers where the subset is an infinite subset correspondingly we may get an infinite sequence and in that case the series will be a infinite sum in short what i would like to convey is the following if sequence a_n is finite the corresponding series is finite if sequence a_n let us write n equal to 1 to infinity is infinite then corresponding series is infinite note that a finite series which may look like $a_1 + a_2 + \dots + a_n$ is just sum of finitely many real values hence it always represent a finite real number whereas an infinite series $a_1 + a_2 + \dots$ may or may not represent a finite real number let us recall the notation we introduce for the sake of simplifying sum namely the sigma notation $a_1 + a_2 + \dots + a_n$ can be written using sigma notation summation a_i i is equal to 1 to n analogously a series $a_1 + a_2 + \dots$ can be represented using sigma notation as follows $a_1 + a_2 + \dots$ is equal to summation a_i i is equal to 1 to infinity this infinity in the upper limit is used to denote that the series that we deal with is infinite series having defined what is a series let us proceed with one more definition consider the sequence a_n let us say n equal to 1 to infinity we construct a new sequence out of the given sequence a_n and that sequence i call as s_n as follows the first term s_1 is same as a_1 the second term s_2 is same as sum of first two terms of the given sequence s_3 is $a_1 + a_2 + a_3$

so on s_n is $a_1 + a_2 + \dots + a_n$

so on s_n can be denoted using sigma notation let us recall you are given with a sequence a_n we construct a new sequence s_n where the terms are defined as follows the sequence s_n is called sequence of partial sums of the series summation a_n n is equal to 1 to infinity we are making the definition of sequence of partial sums remember given a sequence we have corresponding series expressed as summation n equal to 1 to infinity a_n and for this series how do we define sequence of partial sums to define sequence of partial sums we construct a new sequence sequence s_n were s_1 is a_1 s_2 is $a_1 + a_2$ and

so is the definition clear now the question is what is this definition for recall that when we deal with sum of infinitely many real numbers $a_1 + a_2 + a_3 + \dots$ we cannot keep on adding and see what comes out then how do we assign a definite meaning for an infinite sum $a_1 + a_2 + a_3 + \dots$ and

so on the answer is the following we construct sequence of partial sum yes n the n th term namely s_n in the sequence of partial sums is called n th partial sum s_n is called n partial sub remember n th partial sum s_n is $a_1 + a_2 + \dots + a_n$ which can be found by normal addition next what we do is we observe what happens to s_n as n becomes larger and larger that is we observe convergence or divergence of s_n sequences if the sequence of partial sums s_n is convergent

then we say that the series is convergent and if the sequence of partial sum is not convergent we say that the series is divergent let us make it as a definition for a given sequence a_n and corresponding series summation $\sum a_n$ is equal to one to infinity construct sequence s_n called the sequence of partial sums if this sequence of partial sum is convergent that is if there exist a real number l such that as you progress towards the end of the sequence s_n the terms are becoming close enough to this l then we say that summation $\sum a_n$ is convergent otherwise we say summation $\sum a_n$ is divergent intuitively you can understand like this instead of adding a_1 plus a_2 plus a_3 and plus

so on and seeing what comes out which is not practical we first find sum of first n terms that is we find s_n nth partial sum we observe a pattern and see if s_n becomes close to some number l as n becomes larger and larger if yes that number is taken as the sum of this infinite series in notation if $\lim_{n \rightarrow \infty} s_n = l$ meaning the terms of sequence s_n become close to l as n becomes larger and larger then we say summation $\sum a_n = l$ and that l is taken as the sum of this infinite series instead of keep on adding and observing what comes out we observe what happens to partial sum as n becomes larger and larger and the sequence convergence is used to say whether the series is convergent or not given a series we form a new sequence called sequence of partial sums and if the sequence of partial sum is convergent we say the series is convergent intuitively that means this infinite sum ultimately give rise to a finite value when we say a series is convergent what we mean is the series is summable in the sense after adding all these infinite number of real values we will come up with a finite value

so to sum up how do we assign a definite meaning for infinite sum answer is it is through convergence of sequence of partial sums the convergence word is attached with both sequence and series convergence for a sequence is something that says what happens to the sequence as you progress towards tail end of the sequence and convergence of a series is something to say whether the series is summable means after adding all the terms whether you will come up with a finite value or not now we got a definite meaning for an infinite sum a_1 plus a_2 plus etc what does it mean how do we attach a number to it that depends upon whether the series is convergent or not and the series is convergent or not depends upon the convergence of sequence of partial sums may not enter into rigorous study of convergence of sequence and series but then let us have some examples to understand how do we say whether a series is convergent or how do we see whether an infinite sum ultimately give rise to a finite value or not let me give you an example

so the given series is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{16} + \dots$ i hope you can observe a pattern it is powers of 2 in the denominator $2^0, 2^1, 2^2, 2^3, 2^4, \dots$ and

so on just to make the notations clear understand that this series summation $\sum a_n$ arises from the sequence $\frac{1}{2^n}$ n is equal to 0 to infinity first term of that sequence is $\frac{1}{2^0} = 1$ that is the first sum and here second term of the sequence is $\frac{1}{2^1} = \frac{1}{2}$ that is the second sum and in this infinite sum and

so on now the question that we want to answer is whether this infinite sum represents a finite value or not in other words whether this series is summable or not more technical word is whether this series is convergent or not to see this as we developed in the theory first we have to see sequence of partial sum

so let us find sequence of partial sum for this given series the first partial sum namely s_1 is a_1 which is 1 the second partial sum s_2 is $a_1 + a_2$ here it is $1 + \frac{1}{2}$ observe that s_2 can be found by normal addition third partial sum s_3 is $a_1 + a_2 + a_3$ which is $1 + \frac{1}{2} + \frac{1}{4}$ and

so on $s_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ and

so s_n the n to partial sum will be $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}}$

so on to answer the question whether the given series is summable convergent or not we have to see this sequence s_n n is equal to 1 to infinity is convergent or not let us try to observe a pattern see that s_1 is 1 s_2 is 3 by 2 $1 + \frac{1}{2}$ s_3 is 7 by 4 and

so it is bit involved but still with a careful observation it can be seen that there is a pattern and yes n the n th term will be $2^n - 1$ by 2^n n minus 1 first term is 1 second term in the sequence of partial sum is 3 by 2 third term is 7 by 4 and

so n th term is $2^n - 1$ by 2^n n minus 1 it's bit involved to see an expression for s_n like this however let us try to do it in a slightly different manner consider a unit square imagine two copies of unit square are pasted like this the area of this one is one let us have the second unit square area of this first half portion is one by two and second half portion is one by two and the second term in the partial sum s_2 can be attached with the area of first square plus half of the area of the second square

so s_2 is total area which is 2 minus half this portion is missing you can see that s_3 is $1 + \frac{1}{2} + \frac{1}{4}$ which is area of the first disc unit square plus area of half of the second unit square and again you have to add half of the remaining half this is 1 by 4 this is s_3

so s_3 is $1 + \frac{1}{2} + \frac{1}{4}$ which is actually total area minus one by four total area is two minus one by four this portion is missing similarly in s_4 you can see that s_4 is sum of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ that is total area 2 in which 1 by 8 is missing and

so on now you can observe that s_1 is 1 s_2 is $2 - \frac{1}{2}$ s_3 is $2 - \frac{1}{4}$ s_4 is $2 - \frac{1}{8}$ and

so on observing the pattern s_n will be $2 - \frac{1}{2^{n-1}}$ that is what we got s_n is equal to $2 - \frac{1}{2^{n-1}}$ which is $2 - \frac{1}{2^{n-1}}$ thus for a given series namely $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ the sequence of partial sum s_n n is equal to 1 to infinity is given by s_n is equal to $2 - \frac{1}{2^{n-1}}$ observing this sequence of partial sum s_1 s_2 s_3 and

so on it is not hard to see that as n increases terms of the sequence s_n become close to 2 because as n increases this $\frac{1}{2^{n-1}}$ is a large value

so $\frac{1}{2^{n-1}}$ a large value this goes to 0

so as n becomes larger and larger s_n become closer enough to 2 which we write as limit n tending to infinity yes n is equal to 2 .

informally what we have observed is that the sequence of partial sum is convergent and the limit of the sequence of partial sum is 2 .

recalling the definition we made for convergence of a series it should be clear that since the sequence of partial sum is convergent the corresponding series is convergent

so $1 + \frac{1}{2} + \frac{1}{4} + \dots$ is convergent that means this infinite sum ultimately gives you a finite number and what is that number $1 + \frac{1}{2} + \frac{1}{4} + \dots$ is the limit of the sequence of partial sum that is it is 2 thus in this example for a given series we constructed the sequence of partial sum we observed a pattern we could write n th term of sequence of partial sum in terms of n s_n is equal to $2 - \frac{1}{2^{n-1}}$ since we could write s_n in terms of n we could observe what happens to s_n as n becomes larger and larger and we could see that as n becomes larger and larger the terms of the sequence of partial sum comes close to 2 and that 2 is treated as sum of this infinite series if you observe this procedure it should be clear that the success of this depends upon whether we could express s_n in terms of n or not in

this particular example though it is bit involved we could observe that s_n can be expressed in terms of n and through that we could see what happens to s_n as n becomes large enough and through that we could find whether the given series is summable or not however it may not be always possible to find sequence of partial sums in such a manner that s_n is given in terms of n thus testing whether a given series is convergent meaning testing whether given infinite sum finally gives a finite value or not is bit involved task let us not enter to the details of how to check whether a given series is convergent or not etc etc in fact it is not always advisable to find the sequence of partial sum and then check whether the sequence of partial sum is convergent or not we may have to rely on some easy techniques let us not enter into those details to sum up we should have a clear distinction between sequence and series sequence is ordered list of numbers and series is a sum it is sum of terms of a sequence since we may have to deal with sum of infinite number of real numbers and since it is not straight forward as to deal with some of finite number of real numbers we have to reserved on certain notions regarding sum of infinite number of real numbers and the notion is convergence or some mobility and some ability or convergence of series is achieved through convergence of sequence thus the notion of convergence of a sequence help us to break down the narrow confine of sum of finite numbers we can also deal some of infinite number of real numbers using the convergence of sequence next we shall discuss some special kinds of sequence and series where there is some relationship between terms of the sequence or the series in that context we will first discuss what is called as arithmetic progression let us observe some examples consider the sequence 2 4 6 8 10 etc 2n etc in fact it is the ordered list of even numbers consider the sequence 5 10 15 20 25 etc consider the sequence 4 3 2 1 0 -1 etc if you observe the first sequence and ask how the sequence progress we can see that the difference between the second term and the first term namely 4 and 2 which is 2 is same as difference between the third term and the second term 6 minus 4 which is 2.

which is again same as the difference between fourth term and third term similarly in the second example difference between second term and first term is phi which is same as difference between third term and second term which is again phi which is same as difference between the fourth term and third term and so on same is the case with third example where difference between second term and first term namely three minus four is minus one difference between third term and second term namely 2 minus 3 is minus 1 and

so on in these type of examples of sequence the difference between two consecutive terms remains the same such a sequence is called arithmetic sequence or arithmetic progression a sequence in which difference between two consecutive terms remains the same is called an arithmetic sequence or arithmetic progression ap in short in symbols we can define an arithmetic progression as follows a sequence a_n n is equal to 1 to infinity is called arithmetic sequence or arithmetic progression let me write a p if a_{n+1} is equal to $a_n + d$ for every n greater than or equal to 1 where d is a real number n plus one term is obtained from n th term by just adding d this is true for every n

so second term is first term plus d third term is second term plus d and so on where that d remains same here d which is the constant difference between two consecutive terms is called common difference

so in the example we started with two is the common difference in the second example phi is the common difference and in the third example minus 1 is the common difference let us note that a arithmetic progression with first term as a and common difference as d can be written as a $a + d$ $a + 2d$ and

so on thus a $a + d$ $a + 2d$ is the general form of an arithmetic progression with first term a and common difference d observe that when we have

three terms in the ap a , $a + d$, $a + 2d$ there are two common differences that you can work on difference between second and first difference between third and second in fact they will be same however if there are three terms we can work with two common differences if there are five terms you can see that you can work with four common differences if there are n terms there are $n - 1$ common differences that you can think of and the n th term can be obtained by from the first term by adding this $n - 1$ common differences do you see it the sec third term can be obtained from the first term by adding $2d$ and what is that $2d$ two common differences when we move from first term to third term

so in general the n th term of a arithmetic progression with first term a and common difference d is $a + n - 1$ into d when we have to deal with n terms we can have $n - 1$ common differences which when added to a will give the n th term let us observe some facts about arithmetic progression first fact if we add a constant to each term in an arithmetic progression the resulting sequence is again an arithmetic sequence or arithmetic progression that is if we are given with an arithmetic progression a_n n is equal to 1 to infinity in expanded form a_1 , a_2 , a_3 and

so on we construct a new sequence b_n n is equal to 1 to infinity how do we do that b_n is the n th term in the given sequence plus d let me write d dash that is we consider $a_1 + d$, $a_2 + d$, $a_3 + d$ and

so on and the fact is if the given sequence a_1 , a_2 , a_3 is an ap that means the difference of two consecutive terms remains same constant then the new sequence that we constructed remains to be an ap that is the difference between two consecutive terms in this new sequence will also remain the same

so we can construct new aps from the given ap by adding constant with each term same constant to all terms we can continue with some more facts about arithmetic progression and some new types of sequences in the next class thank you you