

welcome to the iit problem solving session on arithmetic geometric and harmonic progressions we are going to have total two sessions on this today we start our session by recalling some facts about them let and be a sequence of numbers we say the sequence a_n is in arithmetic progression in short we say ap if there exists a comma d such that for all n bigger than or equal to 1 we have a_n is equal to $a_1 + (n-1)d$ this d is called the common difference of this arithmetic progression we say this sequence a_n is in geometric progression in short gp if there exists a comma r such that for all n bigger than or equal to 1 we have a_n is equal to $a_1 \cdot r^{n-1}$ we call this r to be the common ratio of this geometric progression and we say the sequence a_n is in harmonic progression in short hp if there exists a comma d such that for all n bigger than or equal to 1 we have $\frac{1}{a_n}$ is equal to $\frac{1}{a_1} + (n-1)d$ that is a_n is equal to $\frac{1}{\frac{1}{a_1} + (n-1)d}$

so basically we say the sequence a_n is in harmonic progression if the sequence $\frac{1}{a_n}$ is in arithmetic progression now we make a note here if we are given three numbers a , b and c which are in an arithmetic progression then often we write a as $b - d$ and c as $b + d$ where d is the common difference of the arithmetic progression if a , b , c are in a geometric progression then we write a as b/r and c as br where r is the common ratio of the geometric progression a , b and c if three numbers a , b and c are in more than one type of progressions then we can derive the following about a , b and c

so our first case is three numbers a , b , c are in ap and as well as in gp since a , b and c are in an arithmetic progression we write a as $b - d$ and c as $b + d$ where d is the common difference of the arithmetic progression a , b and c as they are also in gp we write a as b/r and c as br where r is the common ratio of the geometric progression from here we get ac is equal to $(b-d)(b+d)$ and from here we get ac is equal to $b^2 - d^2$ that means $b^2 - d^2$ is equal to $b^2 - d^2$ and from here we get d^2 is equal to $b^2 - ac$ that is $d^2 = b^2 - ac$

so we get $b^2 - d^2 = b^2 - d^2$

so from here we can conclude that d^2 is equal to 0 that means d is equal to 0 therefore in this case we have a is equal to b is equal to c

so if a , b , c are in ap and in gp then they must be equal our next case is a , b , c are in both gp and hp as a , b , c are in gp we write like in the previous case a is equal to b/r and c is equal to br where r is the common ratio and as a , b , c are also in hp we write a is equal to $\frac{1}{p-d}$, b is equal to $\frac{1}{p}$ and c is equal to $\frac{1}{p+d}$ now from here we get ac is equal to $\frac{1}{(p-d)(p+d)}$ and from here we get ac is equal to $\frac{1}{b^2 - d^2}$

so equating this we get $b^2 - d^2 = \frac{1}{b^2 - d^2}$ note that from here we have $b^2 - d^2$ is equal to 1 therefore $(b^2 - d^2)^2 = 1$ and that implies $b^2 - d^2 = \pm 1$

we have b is not equal to zero therefore d has to be 0 hence in this case a is equal to b is equal to c

so we are getting if three numbers a , b , c are in both geometric progression and in harmonic progression they must be equal our third case is a , b , c are in both ap and hp as a , b , c are in ap we write a is equal to $b - d_1$ and c is equal to $b + d_1$ also as they are in hp we write a is equal to $\frac{1}{p - d_2}$, b is equal to $\frac{1}{p}$ and c is equal to $\frac{1}{p + d_2}$ we see that a is equal to $b - d_1$ as well as a is equal to $\frac{1}{p - d_2}$ therefore equating them we get $b - d_1 = \frac{1}{p - d_2}$ that is $(b - d_1)(p - d_2) = 1$ now from here we see that bp is equal to 1 therefore we get $pd_1 + bd_2$ is equal to $d_1 d_2$ again we have c is equal to $b + d_1$ and c is equal to $\frac{1}{p + d_2}$ we equate them equating them we get $b + d_1 = \frac{1}{p + d_2}$ that is $(b + d_1)(p + d_2) = 1$ as bp is equal to 1 we get $pd_1 + vd_2$ is equal to $-d_1 d_2$ recall that here we obtain $pd_1 + vd_2 = -d_1 d_2$

plus bd^2 is equal to $d_1 d_2$ and now we get pd_1 plus bd^2 is equal to minus $d_1 d_2$ therefore we have $d_1 d_2$ is equal to minus $d_1 d_2$ that means $d_1 d_2$ is equal to 0 that is d_1 is equal to 0 or d_2 is equal to 0 in either of the cases we get a is equal to b is equal to c hence we see that if three terms are in more than one type of progression then they must be equal this is our first question we have three positive real numbers a , b and c and the equation $9a^2 + b^2 + 25c^2 - 3ac$ is equal to $15b + 3a + c$ we are given four options and we shall find out the correct answers we start with the given equation we have $225a^2 + 9b^2 + 25c^2 - 75ac - 45a - 15b - 15c$ is equal to zero note that $225a^2$ is equal to $15a$ whole square $9b^2$ is equal to $3b$ whole square $25c^2$ is equal to $5c$ whole square we can write $75ac$ as $15a \times 5c$ $45a$ as $15a \times 3$ $15c$ as $5c \times 3$

so this whole equation can be rewritten as $15a^2 - 3b^2 + 5c^2 - 15a - 3b - 5c$ is equal to 0.

we can see that all these three numbers are non-negative we know that if the sum of three non-negative numbers is zero then all of them are zero

so we get $15a - 3b$ is equal to 0 $3b - 5c$ is equal to 0 and $15a - 5c$ is equal to 0 that means we have b is equal to $5a$ $3b$ is equal to $5c$ and c is equal to $3a$ now using these we can see that $a + b$ is equal to $2c$ by 3 plus $5c$ by 3 that means $a + b$ is equal to $2c$ from here we can conclude that a , b , c these three numbers are in arithmetic progression

so the second option is correct now looking at the other options we can conclude that none of them are correct now we look at our second question we have 49 numbers a_1, a_2 up to a_{49} which are in arithmetic progression such that sum over k k runs from 0 to 12 $4k + 1$ is equal to $4, 1, 6$ and also $a_9 + a_{43}$ is equal to 66 further is $a_1^2 + a_2^2 + \dots + a_{17}^2$ is equal to $140m$ then we shall find out the value of m note that the first term of the given arithmetic progression is a_1 let d be the common difference of this given arithmetic progression then the n th term of this progression is of the form $a_1 + (n-1)d$ now using this we rewrite this equation we have $a_1 + a_5 + a_9 + a_{13} + \dots + a_{49}$ is equal to $4, 1, 6$ now we keep a_1 as it is then we write a_5 as $a_1 + 4d$ a_9 as $a_1 + 8d$ and

so on and

so forth the last one is $a_1 + 48d$ and this whole thing is equal to $4, 1, 6$ we have $13a_1 + 0 + 4 + 8 + \dots + 48d$ is equal to $4, 1, 6$ as there are 13 terms in this sum

so we get from here $13a_1$ and from the first one we get contribution 0 and from the second one we get contribution 4 and from here we get contribution 8 and continuing like this from the last one we get contribution 48

so therefore we have here $13a_1 + 0 + 4 + 8 + \dots + 48d$ is equal to $4, 1, 6$.

now here there are 12 terms we can rewrite it as taking 4 out $1 + 2$ and the next term was here 12

so this is 3 up to 12

so this is $4 \times 12 \times 13$ divided by 2 which is equal to 13×24

so this equation turns out to be $13a_1 + 13 \times 24d$ is equal to $4, 1, 6$ and taking 13 out we get inside $a_1 + 24d$ and the whole thing is equal to $4, 1, 6$ so finally we get $a_1 + 24d$ is equal to 32.

so this is one equation we have in a_1 and d recall that we are also given $a_9 + a_{43}$ is equal to 66

so therefore $a_1 + 8d + a_1 + 42d$ is equal to 66 that means $2a_1 + 50d$

50 d is equal to 66

so we have another equation in a 1 and d which is a 1 plus 25 d is equal to 33 now we can easily solve these two equations for a 1 and d and we get d is equal to 1 and e 1 is equal to 33 minus 25 is equal to 8 now coming back to the question we shall find out the value of m and we are given a 1 square plus a 2 square up to a 17 square is equal to 140 m i write it here again we have sum over let's say r r runs from 1 to 17 a r square is equal to 140 m now we know what is a r a r is a 1 plus r minus 1 into d and we have already got the value of a 1 is equal to 8 and the value of d is equal to 1

so therefore a r is equal to 7 plus r substituting it here we get sum over r r runs from 1 to 17 7 plus r whole square is equal to 140 m now splitting this we get 49 into sum over r r runs from 1 to 17 here 1 plus 14 into sum over r r is equal to 1 to 17 inside r and the last one is sum over r r runs from 1 to 17 r square is equal to 140 m therefore we have from this part 49 into 17 from this part we have 14 into 17 into 18 divided by 2 from this part we have 17 into 18 into 35 divided by 6 solving this we get m is equal to four seven six zero divided by one four zero that is m is equal to 34 therefore the value of m is 34 and

so the first option is correct this is our third question we have three distinct numbers a b and c if logarithm of a plus c logarithm of a minus c and logarithm of a minus 2 b plus c are in arithmetic progression then among these four given options we shall find out which are correct since log of a plus c log of a minus c and log of a minus 2 b plus c are in arithmetic progression we can write log of a plus c plus log of a minus 2 b plus c whole divided by 2 is equal to log of a minus c that means log of a plus c into a minus 2 b plus c is equal to log of a minus c whole square now exponentiating this equation we get a plus c into a minus 2 b plus c is equal to a minus c whole square from here we get a square plus ca minus 2 a b minus 2 bc plus ac plus c square is equal to a square minus 2 ac plus c square

so a square and a square here gets cancelled c squared and c square here gets cancelled

so finally we get 2 into a b plus bc is equal to 4 a c and this can be written as a b plus bc divided by 2 is equal to ac

so from here we can conclude that a b a c and b c they are in arithmetic progression therefore we see here that the second option is correct now we shall check the other options now among the remaining options note that only one of them can be correct as a b c are distinct numbers here we already got that a b a c b c are in arithmetic progression

so we can write that a b plus bc divided by 2 is equal to ac that is a b plus bc is equal to 2 ac

so we have b is equal to 2 ac divided by a plus c we can write it as 1 by 1 by b is equal to 2 by 1 by c plus 1 by a that is 1 by a plus 1 by c is equal to 2 by b

so from here we can conclude that 1 by a 1 by b and 1 by c are in arithmetic progression that means a b and c are in harmonic progression

so therefore the fourth option is also correct this solves our third question here is our fourth question let a1 a2 up to 18 be in arithmetic progression and h1 h2 up to h10 be in harmonic progression if a1 is equal to h1 is equal to 2 and a10 is equal to h10 is equal to 3 then we shall find out the value of a4 into h7 let us first write what is 8n we know that a10 is equal to a1 plus 10 minus 1

so this is 9 into d where d is the common difference the arithmetic progression a1 a2 up to a10 now we know what is a1 the value of a1 is given to be 2

so we have here 3 is equal to 2 plus 9d because the value of 18 is also here

given which is 3

so we can conclude that d is equal to 1 by 9

so we have found out the common difference of the arithmetic progression a_1 to a_n up to $8n$ we have h_1 to h_{10} they are in harmonic progression that means 1 by h_1 to 1 by h_{10} they are in arithmetic progression let us write what is 1 by h_{10}

so this is 1 by h_1 plus $9c$ where c is the common difference of the arithmetic progression 1 by h_1 to 1 by h_{10} .

substituting the values of h_1 and h_{10} here we get $9c$ is equal to 1 by 3 minus 1 by 2 that is minus 1 by 6

so we get c is equal to minus 1 by 54 .

as we have to find out the value of a_4 into h_7 let us first find out what is a_4 we know a_4 is a_1 plus $3d$

so this is 2 plus 3 divided by 9 therefore this is 7 by 3 and we know that 1 by 8_7 is equal to 1 by h_1 plus 6 into c therefore 1 by 8_7 is equal to 1 by 2 minus 1 by 9

so 1 by 8_7 is equal to 7 by 18

so we have 8_7 is equal to 18 divided by 7 therefore the value of a_4 into 8_7 is 7 by 3 into 18 by 7 and this is equal to 6

so the value of a_4 into eight seven is six and hence the fourth option is correct next we consider the following question in this question we are asked to find out three numbers a , b and c between 2 and 18

so that their sum is 25 the terms 2 , a and b are consecutive terms of an arithmetic progression and the terms b , c and 18 are consecutive terms of a geometric progression let us write down the conditions again

so our first condition is 2 is strictly less than a , b and c and all of them are strictly less than 18 our second condition is $a + b + c$ is equal to 25 our third condition is 2 , a and b are consecutive terms of an arithmetic progression

so from here we can conclude that $2 + b$ divided by 2 is equal to a that means b is equal to 2 into a minus 1 our last condition is b , c and 18 are consecutive terms of a geometric progression

so from here we can conclude that $18/b$ is equal to c^2 substituting b is equal to 2 into a minus 1 here we get c^2 is equal to 36 into a minus 1

so therefore c is equal to 6 into square root of a minus 1 we are considering the positive square root as we know c is positive as now we have the values of p and c in terms of a we substitute them in condition 2 and we obtain $a + 2$ into a minus 1 plus 6 into square root of a minus 1 is equal to 25 that is $3a + 6$ into square root of a minus 1 is equal to 27 we can write it as a minus 9 is equal to minus 2 into square root of a minus 1 now taking square on both sides we get a square minus $18a$ plus 81 is equal to $4a$ minus 4

so a square minus $22a$ plus 85 is equal to zero we can easily note that this implies a minus 5 into a minus 17 is equal to 0 therefore a is equal to 5 or 17 .

let us take our first case to be a is equal to 5 we know that v is equal to 2 into a minus 1 and c is equal to 6 into square root of a minus 1 therefore in case 1 we have b is equal to 8 and c is equal to 12 .

now we can easily note that 5 , 8 and 12 satisfy first and second conditions now we shall check the third and fourth conditions

so consider two five and eight it is easy to note that they are in an arithmetic progression with the common difference 3

so for these values that is for a is equal to 5 , b is equal to 8 and c is equal to 12 the third condition is satisfied now for the fourth condition we consider 8 , 12 and 18 .

we can note that they are also in a geometric progression with the common ratio $3/2$ therefore a is equal to 5 , b is equal to 8 and c is equal to 12 is such a

choice for abc we were looking for next we consider the case a is equal to 17 in this case we can note that p is equal to 32 therefore a is equal to 17 this case is not possible as this choice of a b and respectively if we find out c they fail to satisfy our first condition

so the answer of our question number five is a is equal to 5 b is equal to 8 and c is equal to 12.

now we look at the following question we are given the quadratic equation $x^2 - px + q = 0$ and we are told that alpha and beta are the solutions of this quadratic equation we are also given another quadratic equation $x^2 - 4x + r = 0$ and gamma and delta are given to be the solutions of this quadratic equation if alpha beta and gamma delta are in a geometric progression then we shall find out the integer values of p and q respectively first note that alpha plus beta is equal to 1 and alpha into beta is equal to p as alpha and beta are the solutions of $x^2 - px + q = 0$ similarly we can write gamma plus delta is equal to 4 and gamma into delta is equal to r since alpha beta gamma and delta are in a geometric progression we can write alpha is equal to a beta is equal to ar gamma is equal to ar^2 delta is equal to ar^3 for some a and r here we can write that p is equal to a square into r as alpha is equal to a and beta is equal to ar and q is equal to a square into r to the power 5

so to find out the integer values of p and q it is enough to find out the value of a and the value of r we know that alpha plus beta is equal to 1

so a into 1 plus r is equal to 1 and as gamma plus delta is equal to 4 we have ar square into 1 plus r is equal to 4 we substitute the value of a into 1 plus r in this equation and we obtain r square is equal to 4 that means r is equal to plus or minus 2.

now therefore we get a which is equal to 1 plus r we could write 1 plus r in the denominator as r is not equal to -1

so the value of a is equal to 1 by 3 when r is equal to 2 and minus 1 when r is equal to -2 we also have p is equal to a square into r and q is equal to a square into r to the power five therefore clearly a is equal to 1 by 3 cannot be a possible choice for a as we want b to be integer and q to be integer

so the choice of r is equal to 2 is not possible for us therefore r is equal to minus 2 and a is equal to minus 1

so the value of p is -2 and the value of q is minus 2 to the power 5 that means minus 32

so here the first option is correct this is our seventh question let alpha and beta be the solutions of the quadratic equation $ax^2 + bx + c = 0$ we denote $b^2 - 4ac$ by delta if alpha plus beta alpha squared plus beta squared and alpha cube plus beta cube are in a geometric progression then we shall find out among the four options given here which are definitely true

so first note that since $ax^2 + bx + c = 0$ is a quadratic equation

so a is definitely not equal to 0 and as alpha and beta are the solutions of this quadratic equation we can write alpha plus beta is equal to minus b by a and alpha into beta is equal to c by a also we have the information that alpha plus beta alpha square plus beta square and alpha cube plus beta cube are in a geometric progression therefore we can write alpha plus beta into alpha q plus beta q is equal to alpha square plus beta square whole square we know what is the value of alpha plus beta in terms of a b and c

so let us find out the values of alpha q plus beta cube and alpha square plus beta square in terms of a b and c we know that alpha q plus beta cube is equal to alpha plus beta whole q minus 3 alpha beta into alpha plus beta

so we substitute the value of alpha plus beta and alpha beta here we get this is minus b cube by a cubed plus 3 b c by a square which is equal to 3 a b c minus b cube divided by a cube next let us see alpha square plus beta square we know that this is equal to alpha plus beta whole square minus 2 alpha beta

so therefore this is b square by a square minus 2 c by a that is b square minus 2 ac divided by a square let us now substitute all these values in this equation after substituting v minus b by a into 3 a b c minus b cube divided by a cube is equal to b square minus 2 a c whole square divided by a to the power 4 from both sides we cancel 1 by a to the power 4 and we obtain minus 3 a d square c plus b to the power 4 is equal to b to the power 4 minus 4 a b square c plus 4 a square c square which implies a b square c is equal to 4 a square c square

so we have ac into b square minus 4 ac is equal to 0 as we called b square minus 4ac as delta we get ac delta is equal to 0 now as a is non-zero we can definitely say that c delta is equal to zero therefore we can see that the third option is definitely true but we cannot comment on the rest of the options as we have no information about b and c we end this session here we will resume our next session with problem number eight you