hello and welcome back to the iit pal lectures on mathematics and we have been discussing the binomial theorem and its applications and this is the seventh lecture in the sequence and

so far we have been looking at the very first lecture was the binomial theorem and after that we have been looking at a variety of problems becoming progressively harder right

so today's this this particular lecture is hopefully going to be the last one and given that the difficulty level has been progressively harder today's problems are going to be pretty hard if not i i can't say that they are going to be the hardest of the lot but they are going to be pretty hard ones um actually the iit pal lecture series is supposed to gear you up for a variety of entrance exams after class 12 and these are actual live questions questions that have come questions that might come i don't know ok this is just to prepare you all right

so my first example that i will work on is this

so suppose the question is 2 power k n c 0 times n c k minus 2 power k minus 1 n c 1 n minus 1 c k minus 1 plus 2 power k minus 2 n c 2 n minus 2 c k minus 2 minus etcetera the general term is minus 1 the last term is minus 1 whole power k times n c k n minus k c 0

so this is the this is the expression and this needs to be simplified all right this is the question that you need to simplify this how will you do it

so first of all observe the pattern the pattern is as follows

so if you take any general term out of these many terms any ith term

so this one here i is equal to 0 i is equal to 1 i equal to 2 etcetera all the way till i equal to k right

so if i look at the ith term what do i see i see first of all i see minus 1 whole power i right because the 0th term is positive the first term is negative the second term is positive third is negative etc etc ok

so minus 1 whole power i times 2 to the power k minus i

so the first term is k minus 1 second term is k minus 2 0 term is k minus 0

so 2 power k minus i and then n c i right and then

so this nci nc2 second term first term nc 0 and then what have you got you've got n minus 2 c k minus 2 over here

so that will be for the second term n minus i c k minus i

so this is the general term and if you think about it the net result is nothing but a sigma of this from i equal to 0 all the way till i equal to k ok

so that is the final answer the sigma but let us observe this general term ok you have done expressions where you had the product of two terms n c 0 n c r plus n c one n c r plus one and

so on and

so forth right in that situation this exponent over here was the same as this exponent

so here it is different right here it is nci n minus i c k minus i

so this is not something very straight forward ok not something very straight forward it does not fall into any pattern that you know ok or i have seen before what do you do you are going to expand the nci n minus i c k minus i expand it right if you do not know how to do it expand it

so nci is factorial n by factorial i and factorial n minus i and n minus i c k minus i is factorial n minus i by factorial k minus i and factorial n minus k ok great

so so far

so good now obviously these two cancel out and this is great news because when things cancel out we all feel very happy but you know something it's not that straight forward let's look let's look very carefully over here i have got n

factorial and then i have got n minus k factorial i have got i factorial and i have got k minus i factorial what do you think is needed can i multiply and divide by by k factorial right if i multiply this by k factorial divide by k factorial now what will happen

so ordinarily you like cancelling here not only have i cancelled i have also thrown in something from outside which you didn't expect okay now what is going to happen i have got n factorial and i have got n minus k factorial and i have got k factorial these three are going to combine right how how do they combine i get n c k very good and then i have k factorial i factorial and k minus i factorial and these three will also combine to k c i ok looks much simpler or does it at least i do not have n minus i c k minus i instead i have k c i and the first term is just n c k ok

so now let's rewrite i equal to 0 to k minus 1 whole power i 2 power k minus i times

so what we have simplified

so far is this portion n c k k c i and what is nice is the following this summation is happening when i is vary right as far as this varying thing is concerned n c k is not varying at all n c k is a constant right it's not changing as i is changing

so in all the terms of this summation n c k is a constant

so you can take it common look at the slate of hand that i have produced right i just threw in k factorial in the numerator and in the denominator and lo and behold i now have got something out of which ok

so this is how my expression has simplified great

so i have got nck as common and then i have got a summation how do you tackle this summation now you look at it think about it what's going on let's just write out right if you can't see exactly what is going on let us write it out

so i equal to 0 all the way till k

so i have got k c 0 2 power k times minus 1 to the power 0 plus k choose 1 2 power k minus 1 times minus 1 to the power 1 plus k c 2 two power k minus two times minus one whole power two plus dot dot dot all the way till k right and what is this this is equal to 2 minus 2 plus minus 1 whole power k right that is the binomial expansion of 2 plus minus 1 whole power k isn't it right 2

so k 2 0 you do not choose any minus ones you choose all twos then k choose one you choose only one minus one and you choose k minus one twos and then k choose two you choose two minus ones and k minus two 2's right and take the expansion of this

so this entire summation pretty much boils down to what i have over here and 2 plus minus 1 is nothing but 1

so 1 to the power k is nothing but 1.

so this whole answer is equal to n c k all right was this a nice problem this is a nice problem if you know how to do it right why i am saying this is because of what i got in from outside right it's otherwise a very hard problem if you don't know if you don't realize by inspection that you need to throw k factorial in the numerator and denominator this problem is going to make life miserable for you however with practice right with practice you will be able to see this and understand that you need k on k factorial on the numerator and in the denominator right away just by looking at the break up okay all you need is practice the more you practice the more you will be able to solve questions like this ok

so its an easy problem but its quite hard if you do not know the trick actually all problems are easy till you know the trick all right

so after the last problem let's try to do a similar problem in the same lines

you see the more you do such problems the more you practice the better you will get at it because a lot of these things are not very obvious okay

so i'll try to set up a problem that is very similar to you know questions that actually come in competitive exams like the iit jee and one of the features sometimes what happens is they give a paragraph kind of question

so they set up a paragraph and then they have multiple questions about the paragraphs

so we are going to i am just going to give an example something similar just constructed a similar paragraph kind of an example okay also a lot of times in the iit especially in the iit entrance exams the terminology used is slightly different

so in in the classes in the lecture

so far we have seen ncr right this in our parlance means that you choose r objects out of n right this means factorial n divided by factorial r divided by factorial n minus r right

so a lot of times in the je exam this notation is written like this

so this is just an added piece of confusion but don't get scared this means nothing this only means ncr these are the binomial coefficients okay

so let's try a problem

so the problem is as follows let us define s k is equal to 3 to the power k times 100 choose 0 times 100 choose k minus 3 to the power k minus 1 hundred choose one ninety nine choose k minus one plus three to the power k minus 2 100 choose 2 times 98 choose k minus 2 minus and then plus dot dot dot this is the last term

so its minus 1 whole power k 100 choose k times 100 minus k choose 0.

so this is the setup of the problem it says that s k is defined as this and then he says in the problem he makes a further definition he says that let v k is equal to half to the power k times s k and he is also going to call this as m of hundred comma k

so this is just his definition now do not get scared right this is a function of two variables hundred and k you see there is hundred inside

so that is where the hundred is coming from don't worry about it right now all right now this is the paragraph setup and then he comes to the questions the questions that he throws

so first he says that can you find this the value of this expression ok and then the second question he is asking is can you find m 100 comma 49 plus m hundred comma fifty these are his two questions based on that paragraph all right

so we will try to work out these these two questions ah but first of all what you will notice is that this is very similar to our last problem right

so why we are doing this is just for practice the more you practice the better you will get at it ok

so let us look at s k first

so s k and we will rewrite it because you know we i do not really like this bracket kind of 100 0 i do not like it

so i am just going to rewrite it in my own notation 3 power k 100 c 0 100 c k minus three power k minus one hundred c one ninety nine c k minus one and then dot dot dot minus one whole power k 100 choose k 100 minus k choose nothing all right this is my ex rewriting of their question and then let us look at the general term

so this is a sum of many terms right it is the sum of many terms let us say 3 power i i is ok let us say 3 power k minus i where i goes from 0 all the way till k do you agree

so there is 3 power k 3 power k minus 1 here it is 3 power 0 right
so when i is 0 i get 3 power k then 3 part k minus 1 3 power k minus 2 etcetera etcetera all the way till 3 power k minus k that is 3 power 0 then this times there's a minus it alternates plus minus plus minus
so the 0th term is positive next term is negative and
so on times 100 c 0 100 c 1
so the i th term is 100 c i right times 100 c k then 99 c k minus 1 98
so the i th term is hundred minus i choose k k minus one k minus two k minus i fine
so this is a general term which i have constructed and this is a summation where i is going from 0 all the way till k and just look at it these two are not really equal right which means that i do not have any shortcuts at my disposal and therefore i need to break it up how do you break it up c i hundred c i is factorial hundred by factorial i factorial hundred minus i and hundred minus i c k minus i is factorial hundred minus i divided by factorial k minus i times factorial 100 minus i minus k minus i
so you get 100 minus k and very nicely these two cancel out but then you are left over with factorial 100 factorial 100 minus k and then naturally you say i need a factorial k as well in the denominator
so if i need a factorial k in the denominator then i should have another factorial k in the numerator right and now you regroup
so i have 100 c factorial 100 factorial 100 minus k and factorial k and this is 100 c k and on the other hand i have factorial k factorial i and factorial k minus i
so what is this k c hi very good ok
so that is what i have and then the remaining part can be thrown back in all right
so this is the story
so far now can we work with this
so look at this ah hundred c k this hundred c k is a constant as far as this summation is concerned because in this summation i is changing
so when i is equal to 1 i equal to 2 i equal to 0 100 c k remains exactly what it is because k is a fixed number ok
so therefore you can take it outside the summation
so you can rewrite it as 100 c k times sigma 3 power k minus i minus 1 whole power i k c i where i is going from 0 all the way till k and now tell me what this is what is this boxed thing this whatever i pointed out this sigma 3 power k minus i minus 1 whole power i k c i can you identify this kci means out of k you are choosing i and then you have chosen i minus once right and you have taken k minus i times ah 3 to the power k minus i
so k minus i times you have chosen 3
so it's as if you have got 3 minus 1 3 minus 1 all of these k times right and out of these k minus i times
so there are k k of these and k minus i times you have chosen 3 and i times you have chosen minus 1 right and then how many ways to do this that is k c i and therefore this entire thing boils down to 3 minus 1 whole to the power k okay
so you need to be able to make this deduction look at this and figure out that this is 3 minus 1 whole to the power k indeed ok once we are done with that the next step is very straight forward the next step is that therefore s k is nothing but 100 c k times 3 minus 1 whole power k which is 2 to the power k all right ok
so far
so good now let us go back to our question our question was compute s k s hundred minus k sigma of that from k equal to zero to hundred and all that i

have done
 so far is i have figured out what is s k
 so s k is this and s hundred minus k what is that going to be equal to it is going to be hundred c hundred minus k times two power hundred minus k all right and then if i do a sigma of that this was my question what do i get i get sigma hundred c k 100 c hundred minus k mind you these two are equal 100 c k is equal to 100 c 100 minus k just by the way reminding you times 2 power k times 2 power 100 minus k which is nothing but 2 power 100 and 2 power 100 can jolly well come outside this summation is this familiar now can you do this summation sigma k equal to 0 to 100 100 c k times 100 c 100 minus k if these two are equal they are equal right they are then this is also equal to remember we had done something like this a long time back right a few classes back we had done this even if you don't remember that's okay right the way we had done it see that is the beauty of mathematics you do not always have to remember everything you can work it out its not very hard right all you have to do is remember how to work it out
 so this is what we had done x plus 1 by x whole power 200 ok and then we were looking at the term independent of x in this result right x plus 1 by x whole power 200 the term independent of x in this is the middle term where i have got x power 100 times 1 by x power 100 right that is the middle term and that middle term is 200 choose 100.
 ok now here you expand this whole thing expand this whole thing term by term and then for every x for every x you need to look for a 1 by x over there
 so for example if this breaks up in if you have x bar hundred over here you need one by x per hundred right
 so that is c zero times c hundred plus c 1 times c 99 plus c 2 times c 98 all the way down right and it
 so happens that c ninety nine is equal to c one c hundred is equal to c zero
 so you get back this or you do not even have to get back this is the same as this right hundred
 so k equal to zero hundred c zero hundred c hundred then hundred c one hundred c ninety nine hundred c two hundred c ninety eight and
 so on and
 so forth okay
 so this sigma is nothing but two hundred choose hundred
 so that is your final answer ok
 so this is the first part of our paragraph and then what we are going to do now we are going to try working out the second part of the paragraph and what was the second part of the paragraph the second part of the paragraph was can you compute m of 100 comma 49 plus m of 100 comma 50 where m of 100 comma k is half power k times s k
 so m of hundred comma k is half power k times s k right and i have worked very hard to figure out that s k is hundred c k times two power k and the great news is that half power k cancels out with 2 power k and you are basically left with 100 ck
 so that is m of hundred comma k hundred c k it's nothing much nothing spectacular right it's just to throw you off all right and then what is his question the question is what is m of hundred comma forty nine plus m of hundred comma fifty and now you can do this can't you m of hundred comma k is nothing but hundred choose forty nine plus hundred choose fifty ok how do you do this are you going to start calculating 100 c 49 plus 100 c 50 i hope you do not right this is even the more very powerful calculators will not be able to calculate these large numbers these are very large numbers
 so do not even think about calculating ok there must be a better way look at

them they are adjacent terms in the binomial coefficients right and they remind you of the pascal's triangle right this was your pascals triangle right this is
  so the way it is done is that if i want to find out this particular term this particular coefficient is the sum of these two okay this one is the sum of these two right and these are all ones
  so if i want any term over here i look at the two terms on top and those two will add up to give me that term that is your pascal's triangle which means that hundred c forty nine plus hundred c fifty this is the zeroth layer this is the one layer two eth layer threeth layer fourth layer like this the hundredth layer right if i look at the hundredth layer and pick two terms then the sum of them is going to be the hundred first layer choose
  so if this is forty nine this is fifty then this is going to be fifty ok and of course 101 choose 50 is also equal to 101 choose 51.

  so this is just for your own i mean if required right a lot of times these questions are multiple choice and maybe 101 c 50 is not one of the choices but 101 c 51 is one of the choices then you say 101 c 50 one
  so whatever is required all right
  so this completes uh this particular problem and then we'll go to the next one
  so as you see uh a lot about lot of this these techniques is about practice the more you practice the easier you will be able to reach the easier the easier it is going to be to solve the questions i will give you one more
  so let us look at the following let us do a product of one plus x one plus x plus x squared ok
  so you have got all of these and let's say i multiply them right and after i do this multiplication i expand it up and i write it as a zero plus a one x plus a two x squared plus a three x cube plus dot dot now you have got many questions question one is how many terms are there in this expansion question two is show that coefficients equidistant
  so for example the first term is a 0 and the last term is let's say a capital n x bar capital n
  so a 0 and x n a a n a 1 and a n minus one capital n minus one a two and a capital n minus two
  so these are coefficients that are equidistant are equal and three sum of odd coefficients is equal to sum of even coefficients is equal to n plus 1 whole factorial by two
  so you have to prove these three ah first question you have to answer and the remaining two you have to prove you want to try how many terms are in this expansion
  so it of course it depends on the value of n
  so let us say n is two right if n is two then i have got only the first two terms one plus x and one plus x plus x square right and then i will get a term corresponding to x cube i will get a term corresponding to x squared i will get a term corresponding to x and i will get a term corresponding to 1 right
  so i will get x cube x squared x and one
  so four terms right
  so it is basically this plus this right two plus 3 minus 1 gives me 4 terms is that ok all right let's say there are 3 3 terms 1 plus x 1 plus x plus x squared 1 plus x plus x square plus x cube right
  so you will have the minimum term is going to be with units and the largest term is going to be with x x squared x cube that is x power 6.

  so you are going to go all the way from units to x power six
  so that means you will have seven terms let us say you have got the next one x

bar four right
 so you will go from units all the way till x times x square times x cube times x power 4
 so that is x part 10
 so 11 terms right n such that means how many terms
 so you are going to go from units all the way till x power 1 plus 2 plus 3 x power 1 plus 2 plus 3 all the way till n and that is equal to x power n into n plus 1 by 2.

 so you will start from units x power 0 and reach all the way till x power n into n plus 1 by 2 that means the total number of terms is going to be n into n plus one by two plus one
 so this is the number of terms fine and of course capital n over here i have written it as capital n this capital n is going to be n into n plus one by two very good now can you show that coefficients equidistant from beginning and end can you show that the coefficients equidistant from the beginning and from the end they are equal how will you do that its not too difficult right all you have to do is work out one sequence with x and one sequence with one by x
 so for example if 1 plus x times 1 plus x plus x squared plus times dot dot dot times 1 plus x plus x squared plus dot dot x power n if this is equal to a 0 plus a 1 x plus dot dot dot tool a capital n x power n then what will be one plus x power minus one
 so replace x with x bar minus one that will automatically be a 0 plus a 1 x bar minus 1 a n x power minus n right no surprises there right but then you can simplify this and write it as you can take x bar minus 1 common x power minus 2 comma x power minus n common and then net you are going to get x power capital n times 1 plus x 1 plus x plus x squared or rather x squared plus x plus 1 x power n plus x bar n minus 1 all the way till x square plus x plus 1.
 right and that is going to be equal to x power n times a 0 plus a 1 x plus sir sorry minus m ok and here you will see that a 0 times x power minus n has therefore got to be equal to a n times x bar minus n and a 1 x power minus n plus 1 this this particular term has to be the second last one this last one has to be the first one
 so a zero has to be a capital n and
 so on and
 so forth ok
 so terms equidistant from the beginning and from the end are therefore equal then the last question can you show that the sum of the odd coefficients is equal to sum of the even coefficients is equal to n plus 1 factorial by 2.
 easy stuff how do you work out sum of odd coefficients equal to sum of even coefficients do you remember we did it for the original x plus one whole power n right we had done something we just plugged in x equal to minus 1 here plug in x equal to minus 1 what happens if i plug in x equal to minus 1 the net answer becomes equal to a big 0 right if i just plug in x equal to minus 1 the net answer becomes equal to 0 which means 0 is equal to a 0 minus a 1 plus a 2 minus a 3 plus dot dot dot right and therefore the odd the sum of the odd terms is equal to the sum of the even terms very good but i do not have a value he wants a value the value is also possible in that case plug in x equal to 1 remember we had done this for the basic setup
 so what these questions are trying to do is trying to test your understanding right how did you work out those problems right its a repeat of that
 so plug in x equal to one what do you get you get 2 times 3 next one will be 4 5 6 n plus 1 right that is what you will get if i plug in x equal to 1 and what will i get over here i will get a 0 plus a 1 plus a 2 plus a 3 plus a 4 all the

way till a n now half of them are equal to the other half of them right

so two times the sum of the odd tops

so the sum of the odd terms is equal to this divided by two and what is this this is n plus one factorial

so there you have your answer is there n plus one factorial by two all right

so ah we have basically covered in the last seven lectures ah variety of problems based on the binomial theorem and at the end of the day the binomial theorem always has to be thought of in terms of the basic structure the basic structure is you have got many terms and then you are choosing right to multiply right that's you always have to work with that perception when you work with the binomial theorem ok

so ah we will stop here and we are actually going to extend this to techniques beyond integer n

so in the next class we are going to look at an extension of the binomial theorem this is not an ordinary extension there is a leap of faith and the leap of faith is that the binomial theorem somehow is still going to work even when n is not a natural number

so we are going to work with that and problems based on that starting from the next class ok thank you you