

hello and welcome back to the iit pal series of lectures on mathematics we are working on the binomial theorem and its applications and this is going to be lecture number five now in the last lecture we were looking at a particular problem this was our problem that we were working on  $f(x)$  is equal to  $1 - x + x^2 - x^3 + \dots + x^{16} - x^{17}$  which is equal to  $a_0 + a_1 + a_2 + a_3 + \dots + a_{17}$   $1 + x + x^2 + x^3 + \dots + x^{16} + x^{17}$  and the question was what is  $a_2$  and first we started a brute force you know expansion

so we saw that the first term is equal to  $a_0 + a_1 + a_2 + a_3 + \dots + a_{17}$ .

then the second term  $-x$  was equal to  $a_1 + a_2 + a_3 + \dots + a_{17}$ .

so this was my second relationship and then we looked at  $x^2$  term and that was this does not have an  $x^2$  there is no  $x^2$  here

so i get  $a_2 + a_3 + a_4 + \dots + a_{17}$   $2x^2 + 3x^3 + 4x^4 + \dots + 17x^{17}$  okay

so these were the relationships you can write more and more and more relationships and you are going to find that all of those relationships are going to be quite complicated and hard to work with and then you will have to solve for  $a_0 + a_1 + a_2 + \dots + a_{17}$ .

so if you write for example the 17th equation you will get  $a_{17}$  straight away right in fact you will get  $a_{17}$  to be equal to  $-1$  you can see it by observation  $a_{17}$  is nothing but  $-1$  fine right then you work with you try to find out  $a_{16}$  then you try to find out  $a_{15}$  it's a very long winded process because the question is not what is  $a_{16}$  the question is what is  $a_2$

so you will have to come backwards all the way from  $a_{17}$  to  $a_2$  and you won't have time in an exam this is the wrong way of doing things whenever you see that something is taking too much time then you know that this is not the way the examiner wants me to work it out right is there a shorter method ok then we said let us throw in some calculus at this problem and i wrote it up like this  $f'(x)$  which is  $df/dx$  is nothing but one becomes the derivative of one is  $0 - 1 + 2x - 3x^2 + \dots + 16x^{15} - 17x^{16}$ .

so this is  $f'(x)$  but you also can take the derivative of this  $a_0$  does not have a derivative its derivative is zero  $a_1 + x$  gives me just  $a_1 + 2x$  then i get  $a_2 + 3x^2 + 4x^3 + \dots + 17x^{16}$  okay

so this is also a relationship now if you plug in  $x = 0$  you get back the same second relationship that you had just noticed if you plug in  $x = 0$  but what happens when you plug in  $x = -1$  when  $x = -1$  if you do plug in that second term becomes zero third term becomes zero fourth term becomes zero seventeenth term becomes zero everything becomes zero except for  $a_1$

so  $x = -1$  if i plug in then i get  $a_1$  on this side on the other side i get  $-1 - 2 - 3 - 4 - \dots - 17$ .

so that gives me the value of  $a_1$ .  
you see that

so that was the beauty of the derivative i did not have to come all the way backwards from a seventeen ok

so a one was easy easy enough what about a two you take the second derivative the second derivative gave me  $2 - 6x + 4x^2 - 5x^3 + 4x^4$  plus dot dot dot minus 17 into 16 x bar 15 this was the second derivative and what did i get i got  $2 - 6x + 4x^2 - 5x^3 + 4x^4$  times 2 into 1 plus x plus 4 a 4 times 3 into 1 plus x the whole square plus dot dot 17 a 17 into 16 times 1 plus x bar 15 and now if i plug in x equal to 0 i get back the same original relationship that i wrote a long time back if i plug in x equal to zero right but if you do not plug in x equal to zero zero right you plug in x equal to zero you will get two is equal to something if instead of that you plug in x equal to minus 1 what are you going to get if you plug in x equal to minus one this term cancels all of these cancel all you are left with is two times a two okay which means two times a two is equal to two plus six plus 4 into 3 which is 12 plus 5 into 4 plus six into five

so this was two into one this was three into two just noting it ok and all i did was i plugged in x equal to minus 1 and i got a 2 straight away

so you see the advantage of doing it this way it's a quite an easy problem we spent a lot of time on it but not because we wanted to spend a lot of time on it we i mean once you do these problems a few times you appreciate the beauty of the structure you need to appreciate the structure of the polynomial right and you know similar problems can easily be constructed and then once you see the problem you immediately know that oh i have to do a derivative and then i have to plug in x equal to something need not be minus one

so that is why we went through those motions right

so what is going to be a two finally a two is therefore this whole thing divided by two i need to know how to figure out this ah gigantic sum right its basically two plus six plus twelve twenty six into five is thirty right then the next one is seventeen into six forty two and

so on all the way till seventeen into sixteen ok how do you do this and what is sigma of n squared do you know how to do sigma of n squared one plus n ah one squared plus two squared plus three squared how do you do it do you know what is sigma of n how do you do it write it backwards right add them up right that gives you the result as n into n plus one by two ok

so sigma from two to seventeen though

so that is from one to seventeen from two to seventeen is going to be that less n equal to one to one ok

so this is going to be 17 into 18 by 2 minus 1 this portion and then sigma n squared how do you do that how do you do sigma n square ok

so what is this sum do you know this you should be knowing this if you are preparing for j e the result is n into n plus 1 into 2 n plus 1 by 6 right even if you do not remember the exact numbers you can cross check it you can plug in n equal to 1 for example right if you plug in n equal to one you get one into two into three by six which is one if you plug in n equal to two then its one square plus two squared right that's five your answer should come to be five

so n equal to two

so two into three that six goes away and remaining five

so this is the correct result okay

so we are of course doing from two to seventeen and

so we have to subtract out an extra 1 from the whole story

so our result is going to be 17 into 18 into 2 into 17 plus one two into seventeen is thirty four

so thirty five by six minus one and the minus ones will go away and your net result is going to be 17 into 18 into 35 by 6 minus a half right 35 by 6 minus a

half half is 3 by 6

so net you have got is 32 by 6 and then cancel things out you have got 17 into 3 into 32 and whatever that is 96 into 32 sorry 51 into 32.

right

so this is two a two and then you have got to divide this by a factor of two so your final answer a two should be equal to fifty one into sixteen fifty into sixteen is eight hundred

so this is eight hundred sixteen all right little bit of calculations were required however the solution is not bad it is elegant it did not take as much time as travelling all the way from 17 to 1.

okay

so the next example that we are going to do and here again this is going to be an exercise in elegance how elegant is your solution

so the question over here is suppose you have got 3 plus square root of 5 whole to the power n ok i do not know what exactly n is and the question states that if this is equal to i plus f where i is the largest integer and f is less than one greater than zero right f is between zero and one right

so i is the integer part of three plus root five whole power n and f is the fractional part some part which is less than 1 right

so 3 plus root 5 whole power n i don't know what that answer is maybe that answer is 201.

75

so i is 201 and f is 0.

75 this is the idea ok

so suppose 3 plus root 5 whole power n is i plus f and this is also equal to rho plus sigma where rho is the rational part and sigma is the irrational part i hope you know the difference between rational and irrational yes you do rational rational number is something which can be expressed as an integer divided by an integer an irrational number is absurd for example is an irrational number right

so if you have got root 5 that is irrational 10 times root 5 is also irrational right 17 times root 2 is irrational these are not the only irrational numbers pi e those are also irrational numbers and then there are many many more irrational numbers but in three plus root five whole power n you are going to break it up into pieces right they are going to be integer pieces and then there are going to be pieces which are going to be factors of root five

so if you have got root five in it then thats the irrational part ok

so here he is declaring that rho is the rational part of 3 plus root 5 whole power n sigma is the irrational part of 3 plus root 5 whole power n he wants you to find or rather he wants you to show that rho is equal to i plus 1 by 2 and sigma is equal to i plus two f minus one by two ok

so this is the problem statement the problem statement is 3 plus root 5 whole power n is equal to some integer plus a fraction and it is also equal to a rational number plus an irrational portion rho is the largest rational part and sigma is the irrational part then prove that rho is i plus one by two and sigma is i plus two f minus one by two

so this is our job at hand how will you do this let's look at the following now this i am going to pull out from my hat magicians hat let us look at 3 plus root 5 whole power n and let us also look at 3 minus root 5 whole power n ok

so 3 plus root 5 whole power n is going to have 3 power n plus c 1 times root 5 3 power n minus 1 times root 5.

so i am showing the rational part irrational part plus c two three power n minus 2 times root 5 squared which is 5 plus c 3 times 3 power n minus 3 times 5 root 5 and

so on and  
so forth ok

so if  $n$  is odd then this term will appear here the  $n$ th term will come here if  $n$  is even then the  $n$ th term is going to show up on this column

so depending on let us say  $n$  is odd in that case the  $n$ th term shows up on this column

so  $3$  power  $n$  minus  $3$  becomes equal to  $0$   $3$  power  $0$

so that is  $1$  and then  $5$  power integer part of  $n$  by  $2$  times root five ok and this one is going to be  $c$   $n$  minus one three power sorry just  $3$  and the same  $5$  times  $5$  to the power integer part of  $n$  by  $2$ .

and this is if  $n$  is odd and if  $n$  is even then this term is not this term is going to be on this column and this term will appear here and you won't have the root five at all because  $n$  is even integer part of  $n$  by two is nothing but  $n$  by two in that case ok

so this is generally what's how it's going to expand now if I did  $3$  minus root  $5$  whole power  $n$  what would have happened  $3$  power  $n$  and then over here I would have gotten a minus sign then the next one this term would have been the same because I would have chosen  $2$  of the root  $5$ 's

so root  $5$  squared minus root  $5$  whole squared is  $5$  whereas for this I would have I would have had to choose  $3$  of the root  $5$ 's which gives me minus  $5$  root  $5$ .

so this would have been a minus over here minus over here and what you are going to find is that all of these would have come with an extra minus sign ok so this is the observation right

so in other words if  $3$  plus root  $5$  whole power  $n$  is equal to  $\rho$  plus  $\sigma$  where  $\rho$  is the rational part and  $\sigma$  is the irrational part then  $3$  minus root  $5$  whole power  $n$  is nothing but  $\rho$  minus  $\sigma$  is that ok this seems reasonable ok in which case all you have to do to work out to do

so here what are you doing you are working out  $\rho$  and you are working out  $\sigma$  all you have to do to work out  $\rho$  you have to add these two and divide by two to work out  $\sigma$  you have to subtract these two and divide by two right is that straight forward

so add the two what are you going to get

so you are going to get  $3$  plus root  $5$  whole power  $n$  plus  $3$  minus root  $5$  whole power  $n$  divided by  $2$  that is  $\rho$  is that fine ok now this is also equal to  $i$  plus  $f$  where  $i$  is the integer part and  $f$  is some fractional part ok but  $3$  minus root  $5$  whole power  $n$  what is that going to be equal to

so these are all integers

so you do not have to worry about these and inside of this root five for example right is something more than two right two plus a fraction

so that fraction times  $c$   $1$  times  $3$  power  $n$  minus  $1$  will give you a little fraction some other fraction all those fractions are going to be added and then that will leave out a certain fraction

so there is some fractional part and that fractional part appears only in this set of the second terms in the odd terms right

so the odd terms give you a fractional part the even terms are just pure integers all right

so this is one observation when you do the plus sign

so you if you get some integer for the integer part and then for this fractional part you get some integer plus some blue fraction ok then for  $3$  minus root  $5$  whole power  $n$  what will you get you will get the same integer sorry this should have been in blue in green the same big integer which is the integer part and then over here I will have minus this integer which is the integer part of the odd terms and then I will have a minus the same fraction the same blue dot

ok now these are all integers all right

so you could call this entire thing if this this can be  $i$

so this whole thing is capital  $i$  and this blue dot is small  $f$

so this whole thing is capital  $i$  plus small  $f$  now this integer portion is not really capital  $i$  right it is something minus something its no longer something plus something but maybe it is some some other integer lets call it  $i^2$  ok minus the blue dot which is the fraction now what is the fractional part of this

so if  $i$  have an integer this is my  $i^2$  and  $i$  subtract out a fraction then the fractional part of  $i^2$  minus  $f$  this is this is  $i^2$  suppose this is  $i^2$  and this is your  $f$  all right then the fractional part of this first of all the integer part of this is not really  $i^2$  it is now going to be  $i^2$  minus one and the fractional part is one minus  $f$  right  $i$  am trying to draw it pictorially right

so the integer part is no longer the old integer this minus this it is this minus this and minus an extra one and the fractional part is one minus  $f$  all right

so three minus root five whole power  $n$  will be some other integer  $i^2$  minus one plus a fraction which is nothing but  $1 - f$

so if  $3 + \sqrt{5}$  whole power  $n$  is equal to  $i + f$  then  $3 - \sqrt{5}$  whole power  $n$  will be some other integer  $i^2$  minus one plus a fraction which is  $1 - f$  all right now let us do some jugglery  $3 - \sqrt{5}$  if you think about it  $3 - \sqrt{5}$   $\sqrt{5}$  is a value which is more than 2 but less than 3 root 4 is 2 root 9 is 3 anything between root 4 and root 9 is more than two less than three right

so three minus root five therefore is more than zero but less than one right

so therefore what is  $i^2$  minus one what is this integer portion three minus root five is more than zero less than one three minus root five whole power  $n$  is also going to be more than zero less than one

so what is the integer part the integer part is nothing but zero

so the entire fractional part is one minus  $f$  right one minus  $f$  the entire fractional part is nothing but three minus root five whole power  $n$

so this is the biggest observation over here all right

so  $3 - \sqrt{5}$  whole power  $n$  is therefore  $1 - f$  this is a very important deduction and this will give you the answer

so let us go back where  $i$  was doing this  $3 + \sqrt{5}$  whole power  $n$  is  $\rho$  plus  $\sigma$  which is  $i + f$   $3 - \sqrt{5}$  whole power  $n$  is  $\rho$  minus  $\sigma$  and this is nothing but  $1 - f$  because the integer part is 0 right now you add and subtract you add equation 1 to equation 2 you get  $2\rho$  is equal to  $i + 1$  and therefore  $\rho$  is equal to  $i + 1$  by 2 you subtract 2 out of 1 you get  $i + f$  minus  $1$  plus  $f$

so  $i + 2f$  minus  $1$  and that is equal to  $2\sigma$  which means  $\sigma$  is  $i + 2f$  minus  $1$  by 2.

all right

so what is our biggest observation here our biggest revelation

so  $i$  have got two revelations first of all if  $i$  have  $3 + \sqrt{5}$  whole power  $n$  and that has an integer part  $i$  and a fractional part  $f$  then  $i$  look at  $3 - \sqrt{5}$  whole power  $n$  right the integer part whatever it is it is the fractional part will be  $1 - f$  this is my observation 1 the second observation is that  $3 - \sqrt{5}$  is a number between 0 and 1 which means that if  $i$  take it to the power  $n$  it is also going to be within 0 and 1 which means that this integer part is nothing but 0 there is nothing there all  $i$  have is one minus  $f$  ok

so three minus root five whole power  $n$  itself is the fractional part which is one minus  $f$  all right let's do another one ok this is the next question  $7 + 4\sqrt{3}$  whole power  $n$  that has an integral part  $i$  and the fractional part small  $f$

and you have to show that  $1 - f$  times  $i$  plus  $f$  is equal to 1.

look at this one minus  $f$  very similar in structure right

so you know seven plus four root three what is seven minus four root three whole power  $n$  that will also have an integer part and a fractional part now lets not think about the integer part right now the fractional part of this will be  $1 - f$  minus the fractional part of the original

so if the original is  $f$  then the fractional part of this is  $1 - f$  ok and what is the integer part

so  $7 - 4\sqrt{3}$  is like 1.

7 1.

732 right

so 4 times 1.

7 is how much is that 6.

8 right

so  $7 - \sqrt{3}$  is  $7 - 6$ .

8

so clearly it's something like something less than 0.

2 okay

so it is less than 0.

2 that to the power  $n$  is certainly going to be less than 1 which means that the integer part of this is nothing but 0 which means  $7 - 4\sqrt{3}$  whole power  $n$  is equal to  $1 - f$  all right

so if that is the case if  $7 - 4\sqrt{3}$  is equal to  $1 - f$

so this is  $7 - 4\sqrt{3}$  this is  $7 + 4\sqrt{3}$  whole power  $n$

so ok and whatever  $i$  got 7 times 7 and  $4\sqrt{3}$  times  $4\sqrt{3}$ .

so 49 minus 16 into 3

so 48 ok you have got your answer fine

so this was easy once you get the hang of it it's all really easy let's try another one

so let's say the question is  $p$  is equal to  $2 + \sqrt{3}$  whole power 5 and this  $p$  has an integer part and a fractional part  $f$  is the fractional part

so the way the question can be stated is  $f$  is equal to  $p$  minus the largest integer inside  $p$  ok this is given find  $f$  squared by  $1 - f$  okay

so this is the question find the value of  $f$  squared by  $1 - f$  how do we do this

so once again i look at  $2 + \sqrt{3}$  whole power 5 right it is going to break up into some integer terms some fractional terms the fractional terms are all related to  $\sqrt{3}$  right if i look at  $2 - \sqrt{3}$  whole power 5 then i will get the same integer terms and the same  $\sqrt{3}$  terms but the  $\sqrt{3}$  terms are all going to come with a negative sign which means that the fractional term in this is nothing but  $1 - f$  all right then what about the integer term there is an integer term  $2 - \sqrt{3}$  root 3 is 1.

732 right

so  $2 - \sqrt{3}$  is less than 1 0.

26 something

so that whole to the power 5 is certainly going to be less than 1 which means that the integer term is 0 which means that all you have is the fractional term

so  $2 - \sqrt{3}$  whole power 5 is nothing but is equal to  $1 - f$  where  $f$  is the fractional part of  $p$  all right

so this is what i have

so then what is  $f$  ok and then you have to find  $f$  squared by  $1 - f$   $f$  squared is  $1 - 2 - \sqrt{3}$  whole power 5 the whole squared and  $1 - f$  is nothing but  $2 - \sqrt{3}$  whole power 5 all right and then what are you going

to do how will you solve this how will you work this out you have to rationalize it right

so you multiply numerator and denominator by  $2 + \sqrt{3}$  whole power 5.

ok and  $2 + \sqrt{3}$  times  $2 - \sqrt{3}$  is  $4 - 3$  which is  $1$  whole power 5 is 1

so this whole thing is just one

so you do not have to worry about the denominator and in the numerator you have got  $2 + \sqrt{3}$  whole power five into one minus two into  $2 + \sqrt{3}$  whole power five into two minus  $\sqrt{3}$  whole power five and then plus  $2 + \sqrt{3}$  whole power five into two minus  $\sqrt{3}$  whole power ten but you already know that  $2 + \sqrt{3}$  times  $2 - \sqrt{3}$  is one

so  $2 + \sqrt{3}$  times  $2 - \sqrt{3}$  whole power five is one correct

so  $2 + \sqrt{3}$  whole power five and  $2 - \sqrt{3}$  whole power five will cancel out

so this is what will remain all right and how will you expand this out all the odd terms this has positive odd terms and this will have the same odd terms but negative

so the odd terms do not even have to be considered only the even terms matter if you look at these two only the even terms need to be accounted for

so the even term here is  $2^5$  right into two right let us not write it twice i will just write a 2 into then the next one  $2^4$  will cancel out right then the next one is  $2^3$  and  $5 \times 2$  right what is  $5 \times 2$  five  $\times$  two is ten and  $\sqrt{3}$  squared right and two times that then the third term again will cancel out because its  $\sqrt{3}$  cube right we do not worry about it the fourth term matters what is going to be the fourth term the fourth term is going to be  $2^2 \times 5^2 \times 4$  right  $5^2 \times 4$  is  $5^2 \times 2^2$  and  $\sqrt{3}$  whole power four which is nine okay and then the fifth term is  $\sqrt{3}$  power five minus  $\sqrt{3}$  power five they will cancel out

so you do not have to work with the fifth term this whole thing minus two

so  $2^5$  is what thirty two eight fours are twenty four two hundred forty and this is ninety then what is it two hundred seventy to three hundred sixty two all right not doing it this way would have made you run around in huge circles

so this this problem would have been extremely complicated if you had not done it this way if you had thought of some other complicated method ok want to do another variety of problems sure

so you are given an equation  $x^{2001} + \frac{1}{2} - x$  whole to the power two thousand one is equal to zero ok if you have an equation which is like this right it's uh a polynomial you if once you manage to expand this i cannot expand this right but someone who can expand this will find that it is a polynomial of order 2001 right and that is equal to 0 means that it is going to have how many roots how many solutions will it have it will have two thousand and one solutions ok now not all of not all these solutions are going to be real some solutions are going to be real some other solutions are going to be complex numbers okay but the big question over here is find the sum of the roots right

so which one is real which one is complex is irrelevant we have to find the sum the sigma of all of these groups and figure out what its value is ok how do we do this now imagine all these roots individually right

so let us say that these roots are  $p_1, p_2$  and

so on  $p_3$  all the way till  $p_{2001}$  right suppose these are the roots then this entire expression can possibly be rewritten as  $(x - p_1)(x - p_2) \dots (x - p_{2001}) = 0$  is this reasonable maybe right maybe not we have to check the power of  $x$  to the

power 2001 in this but maybe this is reasonable okay

so a quick check shows that this is unreasonable why because  $x^{2001}$  doesn't really appear no it does it does it does it appears

so this is reasonable but there is going to be some factor over here right

so you have to pick a factor no this is unreasonable you do not get 2001 roots at all

so this will expand into a lot of terms half to the power 2001 plus etc etc all the way till  $x$  to the power 2001 and that will have a minus sign that minus  $x$  to the power 2001 is going to cancel out with plus  $x$  to the power 2001 the net result is going to be that this is not going to be a polynomial of order thousand one it is going to be a polynomial of the order two thousand okay

so if it has if it is a polynomial of order two thousand then it has only two thousand roots in which case this is not going to appear the last term you will see is  $x$  minus  $p^{2000}$  right for the 2000th term what will you get you will get  $x$  to the power 2000 minus  $x$  whole to the power 2000 which is a plus and you will get a half right

so when it comes to  $x$  to the power 2000 the coefficient is going to be half if you expand this expression the coefficient should match which means you should have a half over here all right now this is another way of writing the same story that's what i am declaring

so i am declaring that  $x^{2001} + \frac{1}{2} x^{2000} - \frac{1}{2} x^{2000}$  this whole thing is equal to half times  $x$  minus  $p^1$  times  $x$  minus  $p^2$  all the way till  $x$  minus  $p^{2000}$  ok where  $p^1$   $p^2$   $p^3$  all of these are the roots of the polynomial ok

so what do we do over here one thing you can do is you can try to expand this out and when you try to expand this out the first term

so first of all you have got  $x^{2000}$  right that is by construction and then you have got  $x^{1999}$  times  $p^{2000}$  times  $p^{1999}$  and

so on all of them ok

so the sum of the roots times  $x^{1999}$  right

so the coefficient of  $x^{1999}$  in this expansion is the sum of the roots all right

so all you now have to do is find out the coefficient of  $x^{1999}$  and what is that going to be 2001 choose two all right

so what is your final answer your final answer is  $\sum_{i=1}^{2000} p_i$  is equal to two thousand one choose two which is nothing but two thousand one into two thousand by two all right

so we have done quite a large variety of problems and we can do a few more problems in the next class these are all variants of the binomial theorem you might not be seeing the theorem in action directly every time but what are you doing every time you are choosing a few terms not choosing some terms choosing some terms and rearranging the exponent

so let us stop here and i will see you soon thank you you