welcome to the iit palm series of lectures on mathematics this is a sequence of lectures on on the binomial theorem and its applications and this is lecture four in the last several lectures we have discussed the binomial theorem we have come up with the statement of the theorem we have seen several applications of the binomial theorem and today i am going to focus mostly on problem solving right now lets take our first problem that i wanted to do today um

so in this set of problems there there can be a variety of problems for ah related to the binomial theorem

so one of the concepts is the numerically greatest term now what do i mean by this

so think about this you have got let us say x plus y whole power n and this is equal to x power n plus n choose 1 x power n minus 1 y plus n choose to x power n minus 2 y squared plus dot dot dot all the way till y power n now now ah which of these n plus 1 terms there are n plus 1 terms here just by the way if you did not think about it too carefully there are n plus 1 terms here

so which of these n plus 1 terms is numerically the largest okay

so think about it every term is related to the previous term

so if i call this t 0 t 1 t 2 all the way till t n right you can set up a relationship between any term and its previous term

so for example t 1 is related to t 0 t 1 by t zero is equal to n c e one y by x okay

so it's actually i will write it as nc 1 by nc 0 y by x any pick any term pick the pick the second term and the first term t 2 by t 1 is equal to n c 2 by n c 1 and the second term is larger by a factor y smaller by a factor x ok

so this we know now if you look at these ratios t 1 by t 0 then you look at t 2 by t 1 then you look at t 3 by t 2 then what you are going to find is that these ratios will keep changing will keep moving around and you will get a maximum to this ratio right beyond a certain point this ratio is going to start i am sorry ah beyond a certain point the term is going to become smaller right

so this t 1 by t 0 initially might be more than 1 which means if it is more than 1 then t 1 is more than t 0 if this is less than 1 then t 0 is more than t 1 whatever it is t 2 by t 1 if it is more than one then t two is greater than t one and vice versa right

so what you are going to find is that at a certain point t r plus 1 by t r might suddenly become less than 1 right if such a thing happens then t r is greater than t r plus 1 right and there you declare that t r was the numerically greatest term

so this ratio is going to change continuously its probably probably going to decrease continuously monotonically it is going to decrease or it is going to increase continuously its going to be something which moves monotonically in one direction

so its not going to keep going up and down for example

so lets see find the numerically greatest term in lets say two plus three x whole power nine where x is three by two ok

so what what will you do over here you can imagine this as 2 power 9 plus nc one times two power eight times three x plus n c two times two power seven times 3 x whole squared plus dot dot dot right and then the last term is 3 x whole power 9 ncn i am just trying to do this the long way right there there is a shortcut i mean if you are willing to memorize then you can use the shortcut but let us just see conceptually what is going on over here

so conceptually speaking this is the expansion of the entire question 2 plus 3x whole power 9.

now the rth term over here let us pick the rth term and let us pick the r plus one term all right and t r plus one by t r is going to be what n n is what nine

over here ok this is how it is going to look like 3 x by 2 times n c r plus 1 by n c r now n is 9.
what does this mean n c r plus 1 by n c r factorial 9 by factorial r plus 1 by factorial 8 minus r that is nc 9 c r plus 1 and 9 cr is factorial nine by factorial r and factorial nine minus r and now things are going to cancel okay ah between 9 minus r factorial and 8 minus r factorial most of the terms of all the terms of 8 minus r factorial will cancel out and all that will remain is nine minus r okay and between r factorial and r plus one factorial all the terms of r factorial will cancel out and all that will remain is r plus one
so what you have is ok you have further been given that x is three by two
so you can write this as nine by four all right now take a look at this when r is equal to 0 what is this ratio tr plus 1 by tr r is 0
so this is numerator is 9 denominator is 1
so this is equal to 9 9 into 9 by 4 is certainly more than 1 which means that the first term is larger than the 0th term ok then r equal to 1 8 by 8 in the numerator and 2 in the denominator
so it is a factor of 8 by 2 which is 4 4 times 9 by 4 is 9
so still the second term is 9 times larger than the first term okay
so what happens next this numerator is decreasing this denominator is increasing right
so slowly the numerator is decreasing as r increases numerator decreases denominator increases as r goes up which means that this entire factor goes down as r increases from 0.

so there will be a point where r will suddenly become less than 1 that term is numerically the greatest right after that r will still remain sorry this ratio is still going to remain less than 1 it is going to remain less than 1 after that because the entire ratio is going to keep decreasing as r increases the ratio is going to decrease okay
so we just have to find the point where nine minus r by r plus one times nine by four is less than one what is r such that nine minus r by r plus one times nine by four is less than one and can can we work this out this is much easier to work out
so you have got 4 r plus 4 greater than 81 minus 9 r and then put the r's together
so you have got 13 r on this side and you have got 77 on the other side which means r must be greater than 77 by 13.
what is that eight by thirteen is six right
so six is certainly larger than seventy seven by thirteen correct
so the sixth term is going to be the first term
so r equal to 6 is the first term 6
so r equal to 6 will satisfy this r equal to 7 will satisfy this 8 will satisfy this 9 will satisfy this right we are looking at the first one the first point where t r plus 1 by t r becomes less than 1 that is the edge at which you have got the numerically greatest term
so the numerically greatest term is the sixth term in this case
so you want to check it if you have this if you look at the sixth term what happens nine minus six is three by seven times nine by four
so here this becomes just less than one you have got twenty seven by twenty eight ok
so you are just less than one
so the seventh term is just less than the sixth term that means the sixth term was the largest okay one more example let's try this what is the numerically greatest term in 3 minus 5 x whole power 15 where x is equal to 1 by 5 okay

so once again t r plus 1 by t r what is that equal to that is equal to fifteen c r plus one fifteen c r and in the numerator
so the r plus one term is larger by a factor of minus five x ok smaller by a factor of three ok any questions
so far it's larger by a factor of minus five x smaller by a factor of three this is t r plus one by t r and what is this how does this evaluate this is factorial 15 cancels out right then up you have got factorial r down you have got factorial r plus 1 which means r plus 1 stays in the denominator up you've got 15 factorial 15 minus r is in the numerator right and in the denominator you have got 14 minus r
so 15 minus r stays in the numerator ok now what happens here
so let us take the minus out now as r increases what happens this ratio is always negative ok this ratio is always negative 50 minus 15 minus r by r plus 1 r is going from 0 all the way till 15.

so this ratio is always going to be negative ok sometimes the term is positive next term is negative again the term is positive again the next term is negative positive and negative terms keep alternating but if you are looking for the numerically greatest one you are looking for the number the overall number
so this minus sign is just thrown in there to confuse you right it is just an added gimmick which is confusing you ignore the minus sign don't worry about it because what you are looking for is numerically which term is the largest i mean if i have something like 25 minus 32 plus 43 minus 67 numerically which is largest 67 is the largest one okay
so plus or minus it does not matter
so the minus over here is just thrown in to add a spanner in the works
so you are just going to get confused over here do not be confused do not worry too much about this minus ok
so if you do not worry about the minus then as r increases the numerator keeps going down the denominator keeps going up okay and then what happens then there is going to be a certain point after which this entire ratio is going to become less than 1.

so if you look at r equal to 0 for example r equal to 0 then this is 15 numerator is 15 denominator is 1
so this entire fraction is 15 times 5 x by 3 5 times 1 by 5 is 1 by 3
so 15 by 3 is a factor of 5 okay
so you start with 5
so the first term is 5 times larger than the 0th term okay and then slowly you step the value of r there is going to be an r at which t r plus 1 is going to become smaller than t r which means 15 minus r by r plus 1 times 5 x by three five x by three is really one by three is less than one and what is r in that case right that is what you are looking at
so you solve this 15 minus r is less than 3 r plus 3 and then move things around four r must be greater than twelve which means r must be greater than three
so r equal to four when r is equal to three you will still have t r plus one is greater than t r but when r is equal to four t r plus one is no longer going to be greater than t r ok let us check when r is equal to 3 what happens 12 15 minus r is 12 by 4
so that is a factor of 3 times 1 by 3 that is 1 which means t r plus 1 is equal to tr all right when r is equal to 3 when r is equal to 4 t 5 is eleven by five times one by three
so eleven by fifteen right t five is eleven by fifteen times t six sorry t t

five is eleven by fifteen times t four right which means that t four should be regarded as the largest one all right now what if you had actually paid attention to the minus sign what would have happened in that case

so suppose you say that no no this minus sign means something to me i want to take charge of the minus sign as well in that case what's going to happen your fraction is going to be r minus 15 by r plus 1 times 5 x by 3 which is nothing but 1 by 3.

ok and you are looking at a problem over here because r minus 15 by r plus 1 is negative most of the time ok

so it is almost always satisfying this relationship ok

so you cannot really go with tr plus 1 and tr anymore the sign is alternating all right

so you shouldn't worry about this negative ok let us look at some other problems

so this is something similar to what we had done in our last class more practice ok

so this is somewhat more complicated than the ones that we did in the last class although this is the concept conceptually its similar

so find the term independent of x in this complicated expression now notice the following i have got a 1 over here i have got an x and an x cube which means 1 times the term independent of x will be independent of x x times the term for 1 by x will be independent of x and x cube times the term corresponding to 1 by x cube will be independent of x

so you have to look at the expansion of this and 3 terms in the expansion of this which terms x power 0 1 by x and 1 by x cube

so these are the three terms that you are looking for in this expansion and then you are going to add those terms you are going to add one times the first one one times the second one and two times the third one and that will give you the overall picture ok

so we basically need to look at just this expansion evaluate x power zero evaluate one by x and evaluate one by x cube ok now look at how the terms expand

so you know that you have got a general term is n c r in our case it is 9 c r times 3 x squared by 2 whole power 9 minus r times 1 by 3 x whole power r okay this is the rth term ok

so if you look at x in this i have got x to the power 18 minus 2 r over here and from here i have got x to the power r okay which means that this is how the terms are going to be organized

so i am i will start if r is equal to 0 then i will start with x bar 18 r is equal to 1 i will get x bar 15 then x bar 12 9 6 3 0 minus 3 and

so on

so what all are you looking for you are looking for 0 minus 1 and minus 3 now minus 1 is never going to come this term is just absent right

so its uninteresting i mean you are not going to arrive at x power minus 1 at all in this expansion you will come to x bar 0 and you will come to x power minus 3.

when will you come to x power 0 when r is equal to 6 all right and when will you get x power minus 3 when r is equal to 7.

ok

so what you are now looking at is the sixth and seventh term of this expansion

so the sixth term is nine c six times 3 x squared by 2 whole cube times minus 1 by 3 x whole power 6 this is r equal to 6 term and r equal to 7 term is 9 c 7 3 x squared by 2 whole squared and minus 1 by 3 x whole power 7.

so these are the two terms and then what are you going to do you are going to

multiply the first one by a 1.
and the second one by two x cube and then you are going to add them and that will be your term independent of x okay now what is 9 c 6 9 c 6 is factorial 9 by factorial 6 which is 9 into 8 into 7.
by factorial three which is three into two ok and 9 c7 is what it's 9 into 8 by factorial 2 which is 2 and then 3 x squared by 2 whole cube will give you what 3 cube which is twenty seven x bar six but we know x will all cancel out we will just check verify by two cube which is eight times minus one whole power six is nothing but a plus one
so the minus is not really important in this particular term times 1 by 3 power 6 what is 3 power 6 3 cube is 27 3 into 3 into 3
so let's write it as 27 into 27 into x power 6 and clearly the x bar 6 cancels out neatly and the other one i have got a 2 in front x cube in front 9 c 7 i have written out the expansion and then 3 x squared
so 3 squared that's 9 x power 4 by 2 squared
so that's 4 and then minus 1 whole power 7 is a minus and then 3 to the power 7 that's 27 into 27 into 3 into x power 7 and the x power 7 is going to cancel out because by design we have picked the term independent of x the two terms independent of x all right now you will have to do the arithmetic and work it out 27 cancels out ah this 9 cancels out with the 27 and you get a 3 right 8 cancels out with eight
so you get seven divided by eighteen for the first one and for the second one i have got 4 2 cancelling out with 8 9 and 9 cancelling out with 27 and 3.
and that is it ok
so all i have is 2 and 27 and this is a reasonable fraction to work out right is it reasonable yeah its quite ok i mean you just go to 54 right
so that is how it is
so we have worked out the term independent of x in this massive complicated expression ok
so let us try another one
so this is given to you and then you are told that this f of x also happens to be equal to a 0 plus a 1 into 1 plus x plus a 2 into 1 plus x the whole squared plus a 3 into 1 plus x the whole cube plus dot dot dot all the way till a sub 17 1 plus x whole to the power 17 and that will that will give you the x power 17 term ok
so you are told that a 0 a 1 a 2 a 3 a 17 are such that f of x is equal to this and the same thing is equal to this complicated expression now you are told you are asked what is a two ok how will you do this how will you work this out
so lets make a few observations ah if you look at the 1 over here then the 1 equates to a 0 plus a 1 plus a two plus a three plus all the way till a seventeen ok and that is just the one the first term let us now look at the second term the second term is minus x
so who all are producing a minus a who all are giving you the x term
so a 0 is not i have got a 1 times x right and then i have got 2 a 2 times x 3 a 3 times x all the way till 17 a 17 times x ok and then you can cancel out the x from both sides you will get a one plus two a two plus three a three all the way till seventeen a seventeen is equal to minus one ok is that going to help lets see right maybe what you can do is ok we will lets lets do the next one
so we have done this already what about x squared a 0 is not going to give me an x squared a 1 times one plus x there is no x squared in this but a two times x squared and then a three will give me a right three c two times x squared three c two is nothing but and then a 4 will also give me an x squared sorry ok now lets see what we have over here
so in other words i could have written this as 1 is equal to 7 a 0 plus 1 c 0

times a 1 plus 2 c 0 times a 2 1 is if you choose nothing you get only one way of choosing nothing ok here a 1 plus 2 is really 2 c 1 3 is really 3 c 1 and
so on and likewise a two a two ah your your in front of a two you have got two c two which is a one and then 3 c 2 4 c 2 5 c 2 17 c 2 ok how are these terms moving
so what's the difference between 3 c 2 and 4 c 2 for example 3 c 2 is factorial three by factorial two into factorial one four c two is factorial four by factorial two factorial two five c two is factorial five factorial two factorial three and
so on ok what does that mean that means that if you look at this term and if you look at this term what is the ratio this has increased by a factor of 4 the numerator and the denominator has increased by a factor of 2.
then over here you have increased by a factor of 5 the denominator has increased by a factor of 3.
ok the next term you are going to increase by a factor of 6 decrease by a factor of 4 and
so on all right
so let us write it out slightly differently
so these are just observations i have still not done the problem this might may or may not be relevant to the problem a two three c 2 is factorial 3 by sorry factorial 3 by factorial 2 which is just 3 right and four c two is four by two times that this is what i have and of course you can construct many more relationships for x cube x power 4 and
so on and
so forth the question was what is a two now you have got seventeen unknowns seventeen equations you should i mean one should be able to solve it but its not
so straight forward ok cant really do it that way what will you do
so this is where you need lateral thinking all right
so unlike your mathematics till class 10 your mathematics till class 10 had many different parts you had geometry you had algebra you had arithmetic you had mensuration you had trigonometry right all of these different parts did not talk to each other if you are good at geometry it does not mean that you have to be good at arithmetic if you are good at arithmetic it does not mean that you need to be good at algebra and
so on and
so forth
so they were independent somewhat independent of each other they were completely different sectors a problem from mensuration could not be done with need not be done with geometric or algebra right in your class 10 mathematics in your class 12 mathematics unfortunately this compartmentalization of maths is no longer valid right you have to be able to be agile you have to bring in concepts from complex numbers throw them at binomial you have to be able to bring in your calculus throw it at binomial you should be able to use your binomial in calculus you should be able to use your coordinate geometry in your in calculus and i mean anything goes with anything right coordinate geometry will go with trigonometry suddenly you will find trigonometry inside the binomial theorem it's a mess right you should be able to pick up concepts from here and there and apply it elsewhere
so it is all a mix up ok
so what we are going to do over here in this particular problem is we are going to use a little calculus let us try right let us try using a little bit of calculus now f of x is a polynomial in x okay what do you think is d f by d x i am assuming you know calculus d f by d x is nothing but minus 1 plus 2 x minus 3 x squared plus 16 x power fifteen minus seventeen x bar sixteen and

so on and this happens to be equal to you can do the same thing you can do a derivative of this as well how do you do a derivative of this derivative of a 0 is nothing derivative of a 1 times 1 plus x is a 1 times the derivative of 1 plus x which is 1 plus derivative of a 2 plus a 2 times 1 plus x squared is a 2 2 a 2 times 1 plus x times derivative of one plus x which is one and
so on you get the picture all right and now things have suddenly started to look like our sequences above haven't we now equate the one term
so if you look at the one term now you have got minus one is equal to a one plus two a two plus 3 a 3 plus 4 a 4 plus 17 a 17 okay
so you have got this relationship already have you you have ok can do a second derivative
so this is a shortcut you write this as f prime right you can write this as f double prime
so this is the derivative of the first one and this is we need to do the derivative of the next term as well
so the first term the first equality this f prime is equal to minus 1 plus 2 x etcetera this gave me a derivative which is
so much ok and then i have to work with the second one and that what is the derivative of that
so the derivative of that you need to see it i can see it but for you this was off screen
so let's try it this way
so the derivative of you see what i am doing i am going to do a derivative of this line 2 a 2 times the derivative of one plus x is one three a three times two times 1 plus x plus 4 a4 times three into one plus x whole squared all the way till 17 a 17 times 16 into 1 plus x whole power 15.
all right
so we have done some derivatives over here we have done the first derivative we have done the second derivative now if in the derivative equation you plug in x equal to 0 suppose you plug in x equal to zero then all of this goes away all right all of these terms disappear you are left with minus one which will be equal to a one plus two a two plus three a three plus four a four plus dot dot dot plus seventeen a seventeen and guess what that happens to be this equation okay if on the other hand you had the second derivative and you again plug in x equal to 0 then you start with 2 the rest of the terms are all 0 2 is equal to 2 a 2 plus 6 a 3 ok we didn't really did we have it yeah 2 x squared was equal to 2 a 2 plus 6 a 3.
right plus 12 a4 plus dot dot dot 17 into 16 a17 ok
so we have done it slightly differently all right now again coming back to my question my question was what is a2
so i have developed three relationships
so far these are my three relationships in fact what i did was i had two x squared over here and two a two six a three twelve a four ah four into five right ah all of these will cancel out you will get 16 into 17 a17 okay
so our relationship is this that
so this is one relationship a second relationship that i had was minus 1 is equal to a 1 plus 2 a 2 plus 3 a 3 plus 4 a four ok and a third relationship was one is equal to a zero plus a one plus a two ok
so these are my three relationships you can build more relationships if you want to all right how will you solve them can you eliminate anything can you for example eliminate ah between these last two equations the second and third relationships can you eliminate a 1 to a 17 is it possible well if you take the last relationship and subtract out the second relationship a one does get eliminated ok you do you can eliminate a one out if you subtract ah one from the

other right if you subtract between the first two right you if you take the first relationship and the second relationship then you can subtract one from the other and you can get rid of a2 and you will be left with a1 a3 a4 and
so on the other thing to observe is that the number of terms in each of these relationships is decreasing right the first relationship had all 17 top 18 terms next one has 17 terms the next one has 16 right if you write more relationships you will get lesser and lesser and lesser terms right the 17th relationship will have only one term okay
so that is also something to observe now what can you do to simplify things
so do you want to first find out the 17th relationship then look at the 16th one then look at the 15th one and
so on and
so forth come backwards that would be one way of doing things is there a more elegant solution to this
so let's stop here this basically has been a problem problem solving class and we have been looking at a variety of problems unfortunately this problem is still not complete we are going to keep working on this problem but what we have done
so far is we have used the binomial theorem we have we have kept applying the binomial theorem we have kept getting comfortable with the binomial theorem and we have developed some relationships we have also ah thrown in some ah other things that we know from elsewhere for example from calculus and we have developed on the binomial ah expansion and
so on and
so forth
so we are going to continue with this particular problem in the next lecture and hope to see you soon thank you you