

welcome back everybody this is we were having our lectures on the binomial theorem and its applications and this is the second lecture in the series in the last class we had discussed in detail about what exactly is the statement of the theorem and how we came up with it

so the statement of the theorem is as follows $x + y$ whole to the power n is $\binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n-1} x y^{n-1} + y^n$

so you are choosing two y 's and remaining x 's then you are choosing three y 's and remaining $n-3$ x 's and this sequence this series goes on and on till you end up with $\binom{n}{n-1}$ and then you choose only 1 x and $n-1$ y 's and lastly you choose all n as y 's and no x 's at all ok

so this is the binomial theorem right now this we had proved we had developed intuitively you can also prove it with the method of induction etcetera now $\binom{n}{0}$ and $\binom{n}{1}$ these are all from combinatorics that is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ i hope you know the combinatorics it is factorial n by factorial r and factorial $n-r$ right

so this is the definition of $\binom{n}{r}$ ok

so this is the statement of the theorem and this i did just as a very brief recap now some of the very interesting properties and all lot of these are coming just from the fact that these are combinatorics right one of the properties is that the first term $\binom{n}{0}$ this is the first coefficient and the last coefficient in $\binom{n}{n}$ these are equal why is that why is $\binom{n}{0} = \binom{n}{n}$ that is because factorial n by factorial 0 that's 1 by factorial n minus 0 that's factorial n

so that's equal to 1 and $\binom{n}{n}$ it's again factorial n and here you have factorial n and here you have factorial 0 .

so it is the same result then the next result is that $\binom{n}{1}$ and $\binom{n}{n-1}$ are equal why is that because here you have factorial n factorial 1 factorial $n-1$ the other one your factorial n factorial $n-1$ factorial 1 .

so these are mirror images of each other right in other words you could have just as well written the whole thing as $y + x$ whole power n right instead of writing it as $x + y$ whole power n you could have also written it as $y + x$ whole power n and all of those terms would have been reversed right

so obviously the expression has to be symmetric that is the coefficients from one side have to be equal to the coefficients from the other side okay let's take a look at exactly what i mean

so suppose these are my coefficients from this side and i have these as my coefficients from the other side then this zeroth one will be the same as the n th one number one will be the same as $n-1$ number two will be the same as $n-2$ and

so on right they are all going to be mirror images of each other exactly equal to each other

so if n happens to be odd for example if you are dealing with $x + y$ whole cube right you know what happens you get $\binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3$ that's it something into x cube plus something into x squared and then something into x and then something sorry something into y cube

so these two are equal and these two are equal right

so there are two terms in the middle

so you have got zero one two three these two are terms in the middle they are the middle terms but suppose

so we know these coefficients you know them just by the way suppose you are

doing $x + y$ whole power 4 right what would happen you would get x power 4 times 1 plus 4 times x cube y plus 6 times x squared y squared plus 4 times x y cubed plus 1 times y power 4.

right this is your n choose 0 and choose n choose 1 and choose n minus 1 right and here the two coincide the middle term is one there is only one term in the middle right ok

so this is just an observation the next observation is that the sum of these powers is always equal

so you are doing $x + y$ whole power 4 this is 4^3 plus 1 is 4^2 plus 2 is 4^1 plus 3 is 4 and y power 4 the sum of those powers is the same throughout that's another observation okay ah

so like this let's try to do some some more let's come up with a few more results

so when i write c_0 it actually means n choose 0.

so suppose someone asks you a question what is this sum n choose 0 plus n choose 1 plus n choose 2 all the way till n choose n how will you do it these are all the coefficients of $x + y$ whole power n suppose y is equal to 1 and x is equal to 1.

so let us plug in y equal to 1 and x equal to 1 what do you get you get x bar n is 1 x bar n minus 1 is 1 y is 1 y power n is one

so you get all of these terms right net result is this sum is equal to what one plus one whole power n ok thank you this is a good is a good first exercise try another one let us make x equal to 1 and y equal to minus 1 1 minus 1 whole power n what will you get what is 1 minus 1 0^0 power n is 0 obviously right

so if x is 1 y is minus 1 then $x + y$ whole power n has to be equal to a big 0 ok

so 0 is equal to n choose 0 1 to the power n plus n choose 1 1 to the power n minus 1 times y y is minus 1 plus n choose 2 1 to the power n minus 2 which is 1 times y power 2 y is minus 1.

ok and now you group things together you put all the minuses on one side all the pluses on the other side or you need not do all of that

so you can sim just simplify this and say that this is n choose 0 minus n choose 1 plus n choose 2 minus n choose 3

so it keeps alternating right is that okay this is fine ok

so there are of course two cases over here one case is when n is even the other case is when n is odd

so let's quickly think about them very quickly right if n is even ah let's simplify let's do it all the way till 2 power n sorry 2 n right ah 2 n is always even

so instead of doing $x + y$ whole power n let's do whole power 2 n ok

so this is equal to 0 we know that and that is equal to 2 n choose 0 minus 2 n choose one plus two n choose two and

so on and

so forth right plus two n c_n fine and then you bring the even terms sorry even terms to one side the odd terms to the other side

so you get 2 n c_0 plus 2 n c_1 plus 2 n c_2 sorry two n c_2 n this whole sum will be equal to two n c_1 plus two n c_3 all the odd terms on the other side okay

so on one side i have kept the odd terms on the other side i've kept the even terms and now tell me something what is the sum of all these coefficients all of the coefficients what is the sum what is the sum of all the coefficients from our earlier result what is the sum of all the coefficients its 2 power the

exponent right in our in this particular case our exponent is $2n$

so the sum of $\binom{2n}{0} \binom{2n}{1} \binom{2n}{2}$ all of these the sum all the way till $2n$ is 2 to the power $2n$ right and some of them that is half of them this this and

so on are equal to the sum of the others right you have split it into two parts they are both equally heavy but the sum of them is equal to 2 power $2n$

so what is each each of the sets are 2 power $2n$ minus 1 each of those sets

so $\binom{2n}{0}$ plus $\binom{2n}{2}$ plus $\binom{2n}{4}$ ok this is another very interesting result that we came up with and all of this we are getting just by plugging in some numbers at random right it might seem like i am plugging these numbers in at random but these are nice results to get okay let's try one more another result another very popular result is the following ok this is a very popular result and lets attempt to derive this how will you do this to start with how will you think

so on the right hand side what you see is $2^n \binom{2n}{n}$ what do you think this means by the way these are all should have been $\binom{2n}{0}$ in $\binom{2n}{1}$ ok $2^n \binom{2n}{n}$ means that i probably have to do something to the power $2n$ right i have to work out something whole to the power $2n$ what could it be i want to work something out some x and y whole power $2n$ right and the middle of the coefficients the middle coefficient of this expansion $(x+y)^{2n}$ that coefficient is $\binom{2n}{n}$ right i want to find out this middle coefficient and then this middle coefficient has to be it you what you have to do is you have to show that the middle coefficient is equal to the rest of the stuff how will you do it how you do it the rest of the stuff is to the power n right you are doing you are choosing out of n in the rest of the terms

so clearly those terms you are going to deal with something plus something to the power n right

so think about it the way we are going to do this and you will get this with some practice ok we are going to look at something like $(x+1)^{2n}$ and further we are going to find out the coefficient of the term which is independent of x which does not have an x in that particular expansion

so let's try it out first

so $(x+1)^{2n}$ now the middle coefficient is obviously this right and the middle coefficient is not just the coefficient it is the middle term because when you break up $(x+1)^{2n}$ right and look at the term right in the middle you get something $\binom{2n}{0} x^{2n}$ sorry $\binom{2n}{0} x^{2n}$ plus $\binom{2n}{1} x^{2n-1}$ times 1 by x etc right the term in the middle is $\binom{2n}{n} x^n$ times 1 by x^n and these two politely cancel out okay that's the term in the middle and then of course you have many more terms $\binom{2n}{n-2} x^{n+2}$ times 1 by x^{n-2} and lastly $\binom{2n}{2n-1} x$ times 1 by x right you have got many many terms in this expansion but there is only one term which does not have an x in it that term is the middle one okay it's the complete term that term does not have x in it its independent of x okay now is there any other way of breaking up $(x+1)^{2n}$ any other way yes there is you could break it up as $(x+1)^n (x+1)^n$ right and how do you do $(x+1)^n$ $\binom{n}{0} x^n$ plus $\binom{n}{1} x^{n-1}$ times 1 by x plus $\binom{n}{2} x^{n-2}$ times 1 by x^2 and

so on is the first expansion and the next expansion is ok let's modify this a little bit i won't write it as $(x+1)^n$ i will write it as $1 + x + \dots$ okay i am playing some tricks

so what is the expansion of this second term the expansion of the second term is again $\binom{n}{0} 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \dots$ times $1 + x + \dots$ plus $\binom{n}{2} x^2 + \dots$ times $1 + x + \dots$ plus \dots

so you see why i did that right you see why i did that i did that because these two are going to cancel out with each other when you multiply them right they

will cancel out with each other right now what are we looking for we are looking for the term in this entire product which is independent of x right if I multiply x power n with anything else let's say I multiply this term with this term some x remains ok it is not going to be independent of x the only way it is going to remain independent of x is if I multiply this term with the one right below it

so $\binom{n}{0} x^n$ times $\binom{n}{1} x^{n-1}$ by x^n multiply them it's independent of x the term on the top times the term in the bottom independent of x term in the top times term in the bottom independent of x and

so on and

so forth the way I have written it up and clearly therefore the term which is independent of x is nothing but $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n-1}^2 + \binom{n}{n}^2$ that's what was there okay

so $\binom{n}{n}^2$ and therefore I have proved it

so this is an elegant way of proving this complicated result right $\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2$ is hardly any other way to do it all the other I mean it's this is not a very trivial result this requires a lot of thinking lot of hard work to come up with this without the binomial theorem ok this is fine lot of people use this these results are standard ones which ones am I talking about $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$ lot of people use this as a standard result and $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$ is equal to

so all the even terms sum of all the even terms is equal to the sum of all the odd terms which is equal to 2^{n-1} right this is a standard result and the same here this is also a standard result $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = 2^{2n-1}$ yet another standard result which is very similar to this result very similar ok

so uh we are going to look at a more generalization uh a generalization of the earlier identity that we saw

so what we are going to do is

so the identity is as follows $\binom{n}{r} + \binom{n}{r+1} + \binom{n}{r+2} + \dots + \binom{n}{n} = 2^{n-r} - \binom{n}{r}$ and you have to show that this is equal to $2^{n-r} + \binom{n}{r}$ which is equal to $\frac{2^n}{2^r} + \binom{n}{r}$ and $\frac{2^n}{2^r} + \binom{n}{r}$ okay

so this is what you have to show how do we do this this is just notice that if I plug in r equal to 0 I bet get back the earlier one if I plug in r equal to 0 then I get back the earlier one that was $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$ and that will be equal to $2^n + \binom{n}{n}$ is equal to $\frac{2^n}{1} + \binom{n}{n}$ okay

so this is the same new identity is just a generalization of the older one

so a special case plug in r equal to 0 you get back the original okay

so the older one we how did we solve just recollect we plugged in we tried to do the term independent of x in the expansion of $(x+1)^n$ okay that's what we tried now this time I am not going to try $(x+1)^n$ I am going to try $(x+y)^n$

so let us try $(x+y)^n$

so this remember this is the identity that we are looking forward to uh what I am now going to do is you are looking for this as your answer $2^{n-r} + \binom{n}{r}$ now if you look at $(x+y)^n$ then this is going to expand as $x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + y^n$

so which term will be $2^{n-r} + \binom{n}{r}$ which one are you looking for for example x^n will be a term inside this and that will have a coefficient

of $2^n \binom{n}{r}$ ok but that is not $2^n \binom{n}{r}$ plus r

so this is not it right if i look at the next term then it is $2^n \binom{n}{r+1}$ and then the power of x decreases by 1 and the power of y increases by 1 right this is also not it right the one we are looking for is $2^n \binom{n}{r} x^{n-r} y^r$ ok if you do this expansion it expands into many terms this is just one of the many terms these are all terms inside that now this term happens to be our answer okay the coefficient of this term

so the binomial expansion this binomial expansion is not the only way to expand this right maybe you can also expand it as $(x+y)^n$ times $(y+x)^n$ now why did i flip it you'll soon see why i flipped it right it's not totally obvious but i have flipped it for now one in one case i have written $x+y$ in the other case i have written $y+x$ this is just ah my magic okay

so what can we do we are looking for the coefficient of $x^{n-r} y^r$ in the expansion of $(x+y)^n (y+x)^n$ that's our plan

so lets rephrase the question what is the coefficient of $x^{n-r} y^r$ in $(x+y)^n (y+x)^n$ this is our question now there are many ways to break up $x^{n-r} y^r$ ok many ways to break it up let's say you want let's say i borrow y^r from here from this side then from this side i will have to look at $x^{n-r} y^r$ and always remember the sum of $n-r$ and r the sum of these two has to be equal to n right

so just cross check

so $(x+y)^n$ will have a term $x^{n-r} y^r$ what is the coefficient of that $\binom{n}{r}$ and what is the coefficient of y^r in $(y+x)^n$ $\binom{n}{r}$ ok then let us decrease this by 1 and

so suppose i look for $y^{r-1} x$ and on this side oh i'm sorry this will be $\binom{n}{r-1}$ because you are looking for no x 's ok

so $y^{r-1} x$ times $x^{n-r} y^r$ plus 1 now what is the coefficient of $x^{n-r} y^r$ plus 1 in $(x+y)^n$ and the answer is $\binom{n}{r} + 1$.

right how that's the number of y 's you have chosen and what's the coefficient of $y^{r-1} x$ in $(y+x)^n$ you have chosen only one x

so it is $\binom{n}{r-1}$ then you could also have chosen $y^{r-2} x^2$ and you could have chosen on this side $x^{n-r} y^r$ plus 2.

okay

so the product of these two is once again $x^{n-r} y^r$ so what is the coefficient of $y^{r-2} x^2$ in $(y+x)^n$ you have chosen two x 's

so it is going to be $\binom{n}{r-2}$ and what is the coefficient of this one $x^{n-r} y^r$ plus 2 you have chosen $r+2$ y 's

so this is going to be $\binom{n}{r-2}$ right

so like this you can keep constructing things

so let's write it down

so this will give you $\binom{n}{r} + \binom{n}{r-1} + \binom{n}{r-2} + \dots$ the first term

so this is a shortcut you may or may not write the n sometimes writing $\binom{n}{0}$ is the same as writing $\binom{n}{n}$ a lot of times this is implicit okay

so ah in the question papers for example lot of times they just omit the n

so at that time do not get totally shattered and distressed about it

so this n can easily be omitted its implicit in the expression what what it is what they are talking about

so so what we have got over here from the product of these two is this first

term plus the product of these two gives me this is $nc - 1$ and this is $nc - r + 1$ so that is the second term correct

so like this you are going to keep going keep going and then you are going to go till y power

so the y 's are tracking over here

so $c - 1$ $c - 2$ all the way till $c - n - r$ y power r x power $n - r$ on this side right if i have y power r on this side i need y power n on the other side which means i will have x power 0 .

so how many x 's sorry how many y 's have you chosen over here you have chosen $n - c + n$ y 's and on this side how many x 's have you chosen it will be $n - c + n - r$ so the last term gives you $n - c + n - r$

so the sum of all of these terms is $2 - n - c + n$ plus r x power $n - r$ y power n plus r all right

so this is a complicated identity but it sometimes turns out to be useful all right now we are going to try to solve some problems

so i have a prepared list of problems few few problems that i have prepared and then you know as and when you ask me ah more problems we can solve them

so this is my question one and i have a question two ok and then i have a question three ok

so this is my set of questions 3 questions what is the coefficient of x power 7 in this complicated expression second question what is the coefficient of x power minus 7 in this other complicated expression and the third question is what is the relation between a and b between a and b if these two answers are the same that is the coefficient of x power 7 and the coefficient of x power minus 7 in those two terms in those two expansions if they are equal then what is the relationship between a and b

so these are my three questions let's try to solve this ok ah how will you solve expand right binomially expand the ah first one let us say let us try to do the first one $ax^2 + 1$ by bx^{11} .

so you start from $11 - 0$.

a x^2 whole power 11 right clearly this is not going to give you x power 7 okay plus $11 - 1$ a x^2 whole power 10 times 1 by b x what is the power of x in this term

so x power 20 minus 1

so x power 19

so the first term gave me x power 22 the next term is giving me x power 19

so let's observe what happens what will this give me

so this one is giving me 22 this one is giving me 19 what is this giving me x power 18 minus 2

so that is 16 its not 7 as yet right where will i get $7 - 11 - 3$ will give me x power how much twenty two nineteen sixteen i will get thirteen x part thirteen over here then eleven $- 4$ what will i get x power ten eleven $- 5$ eleven $- 5$ i will get what is the power of x 12 minus 5 will give me 7 right and then the remaining ones will keep going down further

so the only term that is giving me x power 7 is this one and clearly the coefficient of x power 7 is therefore going to be $11 - 5$ times a power 6 by b power 5.

all right great this is my answer to question one then let us try question two what is in my question 2 $a - 1$ by b x^2 whole power 11 slightly twisted there is a minus as well ok and eleven $- 0$ a x whole power eleven what is the power of x here eleven very good minus because there is a minus here what is the power of x here x power ten minus two

so that is x^8 .

so here i have 11 here i have 8 then the next one is going to be a plus 11 c 2 a b right and then i will get x^5 not what i want i want x^7 next one is going to be 11 c 3 and there i will get x^2 next i will get eleven c four and there i will get x^{-1} next eleven c five and x^{-4} and then i will get 11 c 6 and then let us write this out completely $ax^5 + bx^6$ and examine it carefully i have got x^5 i have got x^6

so $5 - 12$ i have got x^{-7}

so it satisfies my requirement

so what is the coefficient this is the term i am looking forward to and therefore the answer to my question 2 is 11 c 6 a power 5 by b power 6.

by the way 11 c 5 and 11 c 6 what are these what is 11 c 5 factorial 11 by factorial five factorial six and what is eleven c six it's the same thing ok so they are equal

so then look at my third question what is the relation between a and b if the first answer is the same as the second answer if the two coefficients are equal what is the relation 11c5 obviously that is already equal to 11 c 6

so i do not have to do anything all that i am saying is a^6 by b^5 is equal to a^5 by b^6 .

which means that which means that a is equal to b a is equal to one by b or b

so let us multiply by b^5 yeah a is equal to one by b great ok

so that is the answer to my third question

so this is just a practice problem ok one more practice problem respectively

so this problem statement says a and b are the coefficients of x^3 in $1 + x + 2x^2 + 3x^3 + \dots + 1 + x + 2x^2 + 3x^3 + \dots + 4x^3 + \dots$ respectively ok then ok this is a trick question a very tricky question how will you solve it how will you solve it this is a very tricky question think about it you might not need the binomial theorem at all right all you need to do is think a little bit after all the binomial theorem is nothing it's just a way of putting things together

so if you apply the same way to put things together you can always keep that in mind that you know n choose 0 where did that come from i split up my $x + y$ whole power n into n terms and then from sum i am taking x from some others i am taking y right and then you make that product

so if you think like that all the time then all of these problems can be very straight forward ok think about it what do we do here $1 + x + 2x^2 + 3x^3 + \dots + 4x^3 + \dots$ what is that this is a very big sum right it is not $x + y$ whole power four what will you do you want to split it up into one and the rest ok thats a possibility lets not do it lets not do it that way at all let us think of it as follows the coefficient of x^3 how will you get the coefficient of x^3 let's do some accounting ok how will you do it you can pick the one here the x here x here x here that will give you x^3 you can pick one here x^2 here x here and one here that will give you x^3 right you can

so basically you can pick the one you can pick the x you can pick x^2 also sometimes but if you pick x^3 over here then the remaining three have to be equal to 1 because you want x^3 okay

so you do your variety of things sometimes you pick one sometimes you pick x sometimes you pick $2x^2$ sometimes you pick $3x^3$ if you pick three x^3 from any one of the terms then the other three terms have to be one right ok

so that is a

so you do all of this accounting and get a all right you want to do it ok how many ways can you pick a one

so we are going to do one from here one of the four terms will be one and three of these ok let us do it like this x^3 you can have $1 \times x \times x$ you could have it as $1 \times x^2$ $1 \times x^2$ and x right you could have it as one one one and x^3 ok if any of the terms is x^3 the remaining three terms have to be one how many ways can you pick x^3 i can pick this one i can pick this one this one or this one four choose one ways to pick the three x^3 term right four choose one ways to pick the three x^3 term right all the other terms have to be a one okay then suppose i pick one of the terms as x^2

so there are four choose one ways to pick the x^2 and then from the remaining three terms i have to pick an x all right and then

so that gives me the $x^2 x$ that kind of a setting and then the third is x and x instead you can say how many ways can i pick one because the remaining three have to be x

so four choose ways to pick four choose one ways to pick an x

so this is a right you work this out this is your coefficient a likewise you have to work out b but guess what b has an extra $4 \times$ power 4 term and that should never come in x^3 you are never going to pick $4 \times$ bar 4.

so it is irrelevant isn't it

so you are saying that to get b you add this which you are never going to pick right you are always going to choose your terms to construct x^3 you are always going to choose your terms between these four you are never really going to go to $4 \times$ power 4 you are never going to pick it

so b is also going to be the same which means that $a - b$ is nothing but 1 sorry 0 ok

so this was a tricky question actually you didn't really have to compute this i computed this just for fun you do not have to do it this was just done as a as an exercise right the principle of doing this is the same as the binomial theorem but we are working with a much larger problem right

so term by term you look at it think about it carefully and work it out you will get this answer okay

so i think we are going to stop here for today and we are going to proceed from this in the next class we are also going to look at some other properties ah some generalizations of the binomial theorem in the next class ok thank you you