

welcome to the iit pull problem solving session on binomial expansions we will have total two sessions on binomial expansions let us begin by recalling a few formulas on binomial expansion then we will solve some problems we begin with the binomial theorem for a positive integer n we have $(a + b)^n$ is equal to $\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$ plus

so on and

so forth $\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$ this can be written as $\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$ note that $(a - b)^n$ is equal to $\sum_{k=0}^n \binom{n}{k} a^{n-k} (-b)^k$ we call the symbol $\binom{n}{k}$ as n choose k as it counts the number of ways of choosing k elements from a collection of n elements let us now note down a few useful formulas for a positive integer n and a non-negative integer r which is strictly less than n we have $\binom{n}{r} + \binom{n}{r+1}$ is equal to $\binom{n+1}{r+1}$ note that $(a + b)^n + (a - b)^n$ is equal to $2 \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$ and also $(a + b)^n - (a - b)^n$ is equal to $2 \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$ these formulas will be very useful for our problem solving session in some of the problems we will have certain series expression in terms of binomial coefficients and we will have to evaluate the given series for this it is often useful to relate the given series with binomial expansion or binomial coefficients of a known expression let us look at the following example for positive integers m and n and a non-negative integer k the sum $\sum_{r=0}^k \binom{m}{r} \binom{n}{k-r}$ is the coefficient of x^k in the binomial expansion of $(1 + x)^m (1 + x)^n$ in this sum whenever r is bigger than n or $k - r$ is bigger than m it is taken to be 0.

the fact that this is the coefficient of x^k in the binomial expansion of $(1 + x)^m (1 + x)^n$ can be verified by writing the binomial expansion of $(1 + x)^m$ and the binomial expansion of $(1 + x)^n$ respectively now with all these let us start solving some problems let us begin with a very basic application of the theory of binomial expansions we had to find out which one among these two numbers $(1 + 1)^{50}$ and $(1 + 1)^{50} - (1 - 1)^{50}$ is bigger we are going to use a very simple trick to solve this we write $(1 + 1)^{50}$ as $(1 + 1)^{50} + (1 - 1)^{50}$ and also we write $(1 + 1)^{50} - (1 - 1)^{50}$.

next we look at what is $(1 + 1)^{50} - (1 - 1)^{50}$ this is equal to $(1 + 1)^{50} - (1 - 1)^{50}$ then we use binomial expansion of these two this is equal to $\sum_{k=0}^{50} \binom{50}{k} 1^{50-k} 1^k - \sum_{k=0}^{50} \binom{50}{k} 1^{50-k} (-1)^k$ now we can note that this is equal to $2 \sum_{k=0}^{50} \binom{50}{k} 1^{50-k} 1^k$ and also k is odd $\binom{50}{k} 1^{50-k} 1^k$ next we open up this summation after opening up the summation we get $2 \sum_{k=0}^{50} \binom{50}{k} 1^{50-k} 1^k$ plus

so on and

so forth up to $\binom{50}{49} 1^{50-49} 1^1$.

now note that $\binom{50}{1}$ is equal to 50

so the first term $2 \sum_{k=0}^{50} \binom{50}{k} 1^{50-k} 1^k$ this becomes 100

to the power 50 and next terms are $2 \binom{50}{3} x^3$ to the power 47 and

so on and

so forth up to $2 \binom{50}{49} x^{49}$ therefore we have obtained $101x^{50}$ to the power 50 minus $99x^{50}$ to the power 50 is strictly bigger than $100x^{50}$ to the power 50 that means $101x^{50}$ is strictly bigger than $100x^{50}$ plus $99x^{50}$ to the power 50

so among these two numbers $101x^{50}$ is bigger next we look at this question for a positive integer n the coefficients of the second and the third and the fourth term in the binomial expansion of $(1+x)^n$ are in an arithmetic progression we have to find out what is n we first write down the binomial expansion of $(1+x)^n$ $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots$

so on and

so forth $\binom{n}{n}x^n$ to the power n in now note that the coefficient of the second term is $\binom{n}{1}$ and the coefficient of the third term is $\binom{n}{2}$ and the coefficient of the fourth term is $\binom{n}{3}$ we know that $\binom{n}{1}$ is nothing but n and $\binom{n}{2} = \frac{n(n-1)}{2}$ and $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ we are told that these three numbers are in an arithmetic progression

so therefore we can write $n + \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)}{2} + 2 \cdot \frac{n(n-1)}{2}$ whole divided by 2 is equal to $n + \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)}{2} + n(n-1)$ now we simplify this since n is a positive integer we can cancel n from both sides of this equation and therefore we obtain $6 + n(n-2) = 6 + n(n-1)$

so we get $6 + n^2 - 2n = 6 + n^2 - n$ is equal to $6 + n^2 - n$

so we have a quadratic equation in n which is $n^2 - 9n + 14 = 0$ now we solve this equation for n and we obtain $n = 9 \pm \sqrt{81 - 56}$ divided by 2

so $n = 9 \pm 5$ divided by 2 that means $n = 2$ or 7 we know that in the binomial expansion of $(1+x)^n$ there are $n+1$ terms

so if $n = 2$ in the expansion we get only three terms

so there is no fourth term therefore here $n = 2$ is not possible

so our answer is $n = 7$

so here we have found out $n = 7$.

next we look at this question we have to find out what is the sum of all the rational numbers in the expansion of $(\sqrt{2} + \sqrt[5]{3})^{10}$ to solve this question we write down the binomial expansion of $(\sqrt{2} + \sqrt[5]{3})^{10} = \sum_{k=0}^{10} \binom{10}{k} (\sqrt{2})^{10-k} (\sqrt[5]{3})^k$ plus $\binom{10}{1} (\sqrt{2})^9 (\sqrt[5]{3})^1$ plus \dots

so on and

so forth $\binom{10}{10} (\sqrt{2})^0 (\sqrt[5]{3})^{10}$ to the power 10 now we can note that in this expansion for a term to be rational we need the power of $\sqrt{2}$ to be an even number and at the same time we need the power of $\sqrt[5]{3}$ to be a multiple of 5 and this is possible only in case of the first term and in case of the last term the first term is $\binom{10}{0} (\sqrt{2})^{10} (\sqrt[5]{3})^0$ that means $1 \cdot 2^5 \cdot 1 = 32$ into $\sqrt[5]{3}$ to the power 0 that means 1

so the first term is 32 now we look at the last term $\binom{10}{10} (\sqrt{2})^0 (\sqrt[5]{3})^{10}$ that means $1 \cdot 1 \cdot 3^2 = 9$ into $\sqrt{2}$ to the power 0 that means $1 \cdot 1 \cdot 3^2 = 9$

that means 3 square which is 9

so the last term is 9

so the sum of all the rational numbers is 32 plus 9 is equal to 41

so our answer is 41 next we look at this question we are given an expression $x + \sqrt{x^3 - 1}$ whole to the power 5 plus $x - \sqrt{x^3 - 1}$ whole to the power 5.

after simplifying this expression we can see that this is basically a polynomial we have to find out the degree of this polynomial to solve this question we first write down the binomial expansion of $x + \sqrt{x^3 - 1}$ whole to the power 5 this is equal to sum running from k is equal to 0 up to 5 $\binom{5}{k} x^{5-k} (\sqrt{x^3 - 1})^k$ and next we write down the binomial expansion of $x - \sqrt{x^3 - 1}$ whole to the power 5 this is equal to sum running from k is equal to 0 up to 5 $\binom{5}{k} x^{5-k} (-\sqrt{x^3 - 1})^k$

so our expression turns out to be i am writing it down here again is equal to 2 into sum running from k is equal to 0 up to 5 and k is even $\binom{5}{k} x^{5-k} (x^3 - 1)^{k/2}$ as k is even we can see clearly that this expression is a polynomial there are total three terms in this expansion the term corresponding to k is equal to 0 the term corresponding to k is equal to 2 and the term corresponding to k is equal to 4 we can note that the monomial corresponding to k is equal to 0 is having degree $5 - 0$

so this is x to the power 5 and here $x^3 - 1$ to the power 0

so this is 1

so the degree is 5 for k is equal to 2 the corresponding monomial x to the power $5 - 2$

so this is x^3 and this is $x^3 - 1$ to the power $k/2$ that is 2 by 2 this is 1

so here the degree is 6.

and for k is equal to 4 we have the degree of the monomial is x to the power $5 - 4$

so x into $x^3 - 1$ to the power $k/2$

so 4 by 2

so this is square

so here this monomial is having degree seven

so finally we get the degree of the polynomial is seven

so here the third option is correct in this question we are given the expression $x + 1$ divided by x to the power $2/3$ minus x to the power $1/3$ plus 1 minus $x - 1$ divided by x minus x to the power $1/2$ and whole raised to power 10 we have to find out the term independent of x in this expression for that let us first write down $x + 1$ divided by x to the power $2/3$ minus x to the power $1/3$ plus 1 as x to the power $1/3$ cube plus one divided by x to the power $2/3$ minus x to the power $1/3$ plus one and now for the numerator we use the formula of a cube plus b cube and therefore we have here x to the power $1/3$ plus one into x to the power $2/3$ minus x to the power $1/3$ plus 1 divided by x to the power $2/3$ minus 6 to the power $1/3$ plus one

so finally we obtain this is equal to x to the power $1/3$ plus one and then next we consider $x - 1$ divided by x minus x to the power $1/2$ we write it as x to the power $1/2$ square minus 1 divided by from the denominator we take x to the power $1/2$ out and then x to the power $1/2$ minus 1 for the numerator we use the formula of square minus b square and then we obtain x to the power $1/2$ plus 1 into x to the power $1/2$ minus 1 divided by x to the power $1/2$ into x to the power $1/2$ minus 1

so finally we have x to the power half plus 1 divided by x to the power half so basically we have 1 plus x to the power minus half therefore the given expression turns out to be x to the power one third plus 1 minus 1 minus x to the power minus half whole raised to power 10

so this is equal to x to the power 1 3 minus x to the power minus 1 by 2 whole to the power 10

so we have been able to write down the given expression in a much simpler form now using this simpler expression we shall find out the term independent of x in this expansion

so we have now the expression x to the power one third minus x to the power minus half whole to the power ten

so if we write down the binomial expansion of this we obtain sum running from k is equal to 0 up to 10.

$^{10}C_k x^{k/3} - x^{-k/2}$ now writing these two terms together we get sum running from k is equal to 0 up to 10 $^{10}C_k x^{k/3} - x^{-k/2}$ choose k into x to the power 10 minus k divided by 3 minus k by 2 this is equal to $^{10}C_k x^{2k/6} - x^{-k/2}$

so this is equal to $^{10}C_k x^{2k/6} - x^{-k/2}$ we write it here in the power of x $^{10}C_k x^{2k/6} - x^{-k/2}$ we have to find out the term independent of x that means we have to find out the coefficient of x to the power 0 in this expansion

so we equate $2k/6 - k/2 = 0$ that is k is equal to 4 so the term corresponding to k is equal to 4 in this expansion is $^{10}C_4 x^{2 \cdot 4/6} - x^{-4/2}$ and this is equal to $^{10}C_4 x^{8/3} - x^{-2}$ into 8 into 7 divided by 4 factorial which is 24

so we get this is equal to $^{10}C_4 x^{8/3} - x^{-2}$ and this is the term independent of x in this expansion therefore here the third option is correct in this question we are given the sum $^{50}C_4 + \dots + ^{56}C_3$ we have to find out the value of this given sum for that purpose let us write down each term explicitly $^{50}C_4 + \dots + ^{56}C_3$ corresponding to r is equal to 6 first which is $^{50}C_3$ next we write the term corresponding to r is equal to 5 which is $^{51}C_3$ and if we continue like this v then the last term is $^{55}C_3$ now let us recall the formula for a positive integer n and a non-negative integer r which is strictly less than n we have $^{n}C_r + ^{n}C_{r-1} = ^{n+1}C_r$ we will use this formula repeatedly using the formula we get $^{50}C_4 + ^{50}C_3 = ^{51}C_4$.

then we use this for $^{51}C_4 + ^{51}C_3$ and we get $^{52}C_4$ then next we get combining these two terms $^{53}C_4$.

when we combine these two terms $^{53}C_3$ and $^{53}C_4$ we get $^{54}C_4$ when we combine these two we get $^{55}C_4$

so finally we have $^{55}C_4 + ^{55}C_3$

so therefore the whole sum turns out to be $^{56}C_4$

so therefore here the fourth option is correct this is our seventh question we have to find out the value of $^{21}C_1 - ^{10}C_1 + ^{21}C_2 - ^{10}C_2 + \dots$

so on and

so forth $^{21}C_{10} - ^{10}C_{10}$ to solve this problem let us combine all the terms with positive sign together and all the terms with negative sign together

so the expression turns out to be $^{21}C_1 + ^{21}C_2 + \dots$

so on and

so forth $^{21}C_{10} - ^{10}C_1 + ^{10}C_2 + \dots$

so on and

so forth $10 \text{ choose } 10$ now we can note that this term is nothing but 2 to the power 10 minus 1 because we can write this term as sum running from k is equal to 0 to up to 10 $10 \text{ choose } k$ into 1 to the power 10 minus k into 1 to the power k minus $10 \text{ choose } 0$ now this is equal to 1 plus 1 to the power 10 minus 1 as 10 to 0 is 1

so finally we get 2 to the power 10 minus 1 next we calculate this term for that we write the term $21 \text{ choose } 1$ plus $21 \text{ choose } 2$ plus up to $21 \text{ choose } 10$ as half into 2 into $21 \text{ choose } 1$ plus 2 into $21 \text{ choose } 2$ plus up to 2 into $21 \text{ choose } 10$ now note that $21 \text{ choose } 1$ is same as $21 \text{ choose } 20$ and $21 \text{ choose } 10$ is same as $21 \text{ choose } 19$ and

so on and

so forth $21 \text{ choose } 10$ is equal to $21 \text{ choose } 11$.

so this sum turns out to be half into $21 \text{ C } 1$ plus $21 \text{ C } 2$ plus

so on and

so forth up to $21 \text{ C } 20$ now we add and subtract half into $21 \text{ choose } 0$ and $21 \text{ choose } 21$

so therefore we get this part is equal to one plus one to the power 21

so this whole expression becomes half into 2 to the power 21 minus $21 \text{ choose } 0$ is equal to 1 and $21 \text{ choose } 21$ is also equal to 1

so therefore this is equal to 2 to the power 20 minus 1 therefore our expression is equal to 2 to the power 20 minus 1 minus 2 to the power 10 plus 1

so this is equal to 2 to the power 20 minus 2 to the power 10

so here the first option is the correct answer in this question we are asked to find out the value of $20 \text{ choose } 0$ minus $20 \text{ choose } 1$ plus $20 \text{ choose } 2$ minus

so on and

so forth plus $20 \text{ choose } 10$ let us call this number as you all right now recall that the binomial expansion of 1 minus x whole to the power 20 is equal to $20 \text{ choose } 0$ minus $20 \text{ choose } 1$ into x plus $20 \text{ choose } 2$ into x square plus $20 \text{ choose } 10$ into x to the power 10 minus $20 \text{ choose } 11$ into x to the power 11 up to $20 \text{ choose } 20$ into x to the power 20 now let us put x is equal to 1 in this binomial expansion therefore we get 0 is equal to y minus $20 \text{ choose } 11$ plus $20 \text{ choose } 12$ up to $20 \text{ choose } 20$.

so we have y is equal to $20 \text{ choose } 11$ minus $20 \text{ choose } 12$ plus up to minus $20 \text{ choose } 20$ now note that $20 \text{ choose } 11$ is equal to $20 \text{ choose } 9$ and $20 \text{ choose } 12$ is equal to $20 \text{ choose } 8$ if we keep on writing like this up to the last term we again this is $20 \text{ C } 0$

so we have y is equal to $20 \text{ C } 9$ minus $20 \text{ C } 8$ plus $20 \text{ C } 7$ up to $20 \text{ C } 0$ we add and subtract $20 \text{ C } 10$ here now we write taking minus common here minus of $20 \text{ C } 10$ minus $20 \text{ C } 9$ plus $20 \text{ C } 10$ up to plus $20 \text{ C } 0$ and we have here plus $20 \text{ C } 10$ note that this inside expression is nothing but y

so therefore we have $2y$ is equal to $20 \text{ C } 10$ hence y is equal to half into $20 \text{ C } 10$ therefore here the fourth option is the correct answer in this question we are asked to find out the value of the sum $30 \text{ C } 0$ into $30 \text{ C } 10$ minus $30 \text{ C } 1$ into $30 \text{ C } 11$ plus $30 \text{ C } 2$ into $30 \text{ C } 12$ minus

so on and

so forth plus $30 \text{ C } 10$ into $30 \text{ C } 30$ we rewrite the sum as $30 \text{ C } 0$ into $30 \text{ C } 20$ as $30 \text{ C } 10$ and $30 \text{ C } 20$ are same minus $30 \text{ C } 1$ into $30 \text{ C } 19$ as $30 \text{ C } 11$ and $30 \text{ C } 19$ are same and if we keep on doing like this we get the last term is $30 \text{ C } 20$ into $30 \text{ C } 0$ now note that this sum is nothing but the coefficient of x to the power 20 in the binomial expansion of 1 plus x to the power 30 into 1 minus x to the power 30 now we know that 1 plus x to the power 30 into 1 minus x to the power 30 is equal to 1 minus x square whole to the power 30 and the binomial expansion of

this is equal to sum where k runs from 0 to up to 30 .

30 choose k into minus 1 to the power k into x to the power $2k$
so from here we can see clearly that the coefficient of x to the power 20 is equal to 30 c 10 into minus 1 to the power 10

so basically this is 30 c 10 .

therefore the value of the given sum is 30 c 10 and

so the first option is correct here in this question we are asked to find out the sum of the coefficients of the integral powers of x in the binomial expansion of 1 minus 2 square root of x whole to the power 50 and to solve this problem we first write down the binomial expansion of 1 minus 2 square root of x whole to the power 50 this is nothing but sum over k k runs from 0 to up to 50 50 choose k into minus 1 to the power k into 2 to the power k into x to the power k by 2 now from here we can clearly note that the terms in the sum corresponding to k even are the terms having integral powers of x

so basically we have to find out the value of the sum 50 c 0 plus 50 c 2 into 2 square plus 50 c 4 into 2 to the power 4 plus

so on and

so forth up to 50 c 50 into 2 to the power 50

so the value of this sum is our desired answer now let us note that 1 plus $2x$ whole to the power 50 plus 1 minus $2x$ whole to the power 50 is equal to sum over k k runs from 0 to up to 50 50 choose k 2 to the power k into x to the power k plus sum over k k runs from 0 to up to 50 50 choose k minus 1 to the power k into 2 to the power k into x to the power k and this is equal to 2 into sum from k is equal to 0 to up to 50 and k even 50 c k 2 to the power k into x to the power k by putting x is equal to 1 here we get 50 choose 0 plus 50 choose 2 into 2 square plus 50 choose 4 into 2 to the power 4 plus

so on and

so forth up to 50 choose 50 2 to the power 50 is equal to 1 by 2 into 3 to the power 50 plus minus 1 to the power 50 .

so basically we have 3 to the power 50 plus 1 divided by 2 .

so here the second option is the correct answer this is all for our first session on binomial expansions i end it here you