

in the last few lectures i have introduced the concepts of counting we had the fundamental principles of counting such as addition principle the multiplication principle and then the number of ordered permutations that is ordered subsets and ordered arrangements and the combinations that is the number of unordered arrangements taking from a certain set or certain number of ah distinct items ah today i will solve firstly a few problems on ah this topics and then we will introduced a few more concepts in ah combinatorics

so let me start with some of the problems ah let us take this problem this is from one of the question papers of joint entrance examination two thousand fourteen

so let  $n$  be an integer take  $n$  ah distinct points on a circle and join each pair of adjacent points by blue and the rest by red

so if the number of red and blue line segments are equal then find the value of  $n$

so let me read the problem again we take  $n$  distinct points on a circle and the adjacent pair of points we join by blue line segment and the remaining one we join by red and suppose these red and blue line segments are same in number then what is the value of  $n$  let me ah describe through a diagram

so suppose we consider a circle and there are points on it

so this point this point and

so on

so we have points here now this adjacent points we are joining by ah blue

so like this will be joined by blue this will be joined by blue this will be joined by blue and

so on

so these are all adjacent they will be joined by blue ah like that you can keep and the remaining one we join by red

so let me use a red marker here and

so we can join this by red this will be joined by red this will be joined by red etcetera

so these are all that means if they are not adjacent then we join them by red now let me do the elementary counting here

so if there are  $n$  points then if you join them there will be exactly  $n$  line segments

so there are  $n$  line segments joining adjacent points

so the number of blue line segments that is  $n$  now if we join all the points if we join all the points the total number of line segments ok

so how will you count that there are  $n$  points and if i take every two pair

so that means  $n C 2$  because if you join a with b then it is same as joining b with a

so it is counted twice

so it is basically the number of unordered arrangements of 2 taken from  $n$

so the total number of line segments will be  $n C 2$  that is  $n$  into  $n$  minus 1 by 2.

so we have said that adjacent ones are  $n$  and total number is  $n$  into  $n$  minus one by two

so the remaining line segments are how many  $n$  minus that is  $n$  into  $n$  minus one by two minus  $n$

so we just write that thing

so the number of line segments which are colored red that is  $n$  into  $n$  minus one by two minus  $n$  now it is given that  $n$  into  $n$  minus 1 by 2 minus  $n$  is equal to  $n$

because it is written that the number of red and blue line segments are equal

so we get this equation here  $n$  into  $n$  minus 1 by 2 minus  $n$  is equal to  $n$

so of course you can simplify this it is equivalent to  $n$  into  $n$  minus one is equal to this you take to this side

so it becomes four  $n$

so this means that  $n$  square is equal to five  $n$

so  $n$  must be equal to five that means if there are five points which are chosen on a circle and if we join the adjacent points with the blue and non adjacent points with red and if the number is same then there will be exactly five vertices or five points on the circle

so this is a simple counting problem here we have made use of the combination that is  $n C 2$  ah let me solve another problem which is again from ah one of the je question papers it is again counting in a particular way

so let  $n_1$  less than  $n_2$  less than  $n_3$  less than  $n_4$  less than  $n_5$  be positive integers such that  $n_1 + n_2 + n_3 + n_4 + n_5$  is equal to twenty

so five integers are chosen in such a way that their sum is twenty in how many ways we can choose  $n_1$   $n_2$   $n_3$   $n_4$   $n_5$  in how many ways  $n_1$   $n_2$   $n_3$   $n_4$   $n_5$  can be chosen that means satisfying these criteria now we can start from the minimum

so if we consider say assignment one to  $n_1$   $n_2$  is equal to two  $n_3$  is equal to three and four is equal to four now you see what i have done i have chosen the smallest possible values of  $n_1$   $n_2$   $n_3$  and  $n_4$

so now the remaining choices for  $n_5$  now the sum of ah  $n_1$  and two  $n_3$  and four is ten and the total is twenty therefore it is mandatory that  $n_5$  has to be ten

so now we make little bit flexibility we take another option smallest  $n_1$  is smallest  $n_2$  is smallest  $n_3$  now in  $n_4$  we have already taken the smallest one

so now what is the next one

so the next smallest can be five if we do that then  $n_5$  becomes nine because we have increased this by one

so this has to be decreased by one now let us take another option we still keep  $n_1$  and two  $n_3$  to be the smallest and i again enhance  $n_4$  by one that is we make it six then this will become eight ah now you can see with the smallest  $n_1$   $n_2$   $n_3$  these are the only possibilities of  $n_4$   $n_5$  because in the next step suppose i make  $n_4$  to be seven then  $n_5$  will also become seven

so which will violate this condition that  $n_4$  is less than  $n_5$

so with the smallest value of  $n_1$   $n_2$  and  $n_3$  we have three possibilities now let us take smallest  $n_1$  and two and with  $n_3$  i take the flexibility

so smallest  $n_1$   $n_2$  is one two and then with  $n_3$  in place of three i take four that is the next one

so now let us take the next smallest for  $n_4$  that is five

so now how many ah values we have exhausted that is twelve

so  $n_5$  can be only eight that is the only possibilities now with the same one if i choose second value for  $n_4$  that is one two four and this five i make six that is the next smallest then how many values are ah exhausted

so we have taken seven plus six thirteen

so  $n_5$  can be only seven to make  $n_1$  plus  $n_2$  plus  $n_3$  plus  $n_4$  plus  $n_5$  is equal to twenty again you can see that these are the only possibilities if i choose  $n_1$  and  $n_2$  to be the smallest and  $n_3$  to be the next smallest that is from 3 we have taken to the 4 value no other possibility is there now let us take smallest  $n_1$  and second smallest  $n_2$  that means in place of two i make it three now let us take if i take the smallest  $n_3$  that is four smallest  $n_5$  that is  $n_4$  that is five

so now you have thirteen

so the next value can be only seven that is the only possibility now in the next one if i make this to be six then this will also become six

so that that will violate  $n_4 < n_5$

so with one three that is the smallest  $n_1$  and the next smallest value of  $n_2$  i have only one option now let us take the next one

so now we have exhausted all the options in which  $n_1$  is the smallest in the first one i took  $n_1$  and two and three and four to be the smallest in the next one i took  $n_1$   $n_2$   $n_3$  to be the smallest and and four with the next smallest and the same thing is in the third arrangement also ah then we took  $n_1$  and  $n_2$   $n_3$  to be the smallest and for  $n_3$  i took the next smallest

so four value was taken with that again we can see that there are three possible arrangements now since here only in the last one only  $n_1$  is the smallest and all others are slightly bigger than the smallest therefore the next option should be to go for the second value of  $n_1$

so second value of  $n_1$  we can take to be two and then we take three four five and six as the values for  $n_2$   $n_3$  and four and  $n_5$  you can see easily that this sum is directly equal to twenty therefore now there is no other possibility of increasing any other number that means i cannot make any of the  $n_2$   $n_3$  and four and five to be any other choice other than this

so this is exhausting all the possibilities of choosing  $n_1$   $n_2$   $n_3$   $n_4$  and  $n_5$  in such a way that  $n_1 < n_2 < n_3 < n_4 < n_5$  and the sum of all the integers is equal to twenty

so we have total number of ways that is equal to seven total saving arrangements are there ah

so the purpose of this problems is that we can actually solve or you can say find solutions in the integers these are normally number theory called partition problems are the diophantine equations etcetera

so basically here also counting is required sometimes we use the permutation combination and sometimes direct counting is required let me give one or two other problems also of the similar nature ah let  $x$   $y$   $z$  be integers satisfying system of homogeneous equations  $3x - y - z = 0$   $3x + z = 0$   $x + 2y + z = 0$  find the number of such points for which  $x^2 + y^2 + z^2 \leq 100$  ah

so let us observe if we look at this equation number say one and two and i add these two then i will get only  $y = 0$  or if i take say ah second and third and i subtract then i get  $y = 0$

so immediately it is clear that  $y$  is zero

so here  $y$  is zero now if you put  $y = 0$  then we get  $z = 3x$

so you actually have an infinite number of solutions in general if we can allow all the values of  $x$  and  $z$  but here we have the condition that they are integers as well as the sum of squares has to be less than or equal to hundred now here since  $y = 0$  you are having only the condition that  $x^2 + z^2 \leq 100$

so what are the pairs here ah you are having  $z$  as three times  $x$

so only possible pairs for  $x$   $y$   $z$

so they are not pairs basically ah three tuples for  $x$   $y$   $z$  they are

so obviously zero zero zero is one solution and then you can have  $x = 1$   $z = 3$

so  $z$  will become three and whatever value you take you can take the negative of

that also

so minus one zero minus three will also be one of the solutions and then you can have two zero six

so therefore minus two zero minus six and three zero nine and therefore minus three zero minus nine ah let us look at the sum of squares here with the last one it is becoming three square is nine plus nine square is eighty one

so it is ninety the next one if i take four

so four zero twelve now twelve square becomes one hundred forty four

so that will violate this condition  $x^2 + y^2 \leq 100$  to hundred

so these are the only possible solutions

so the total number of solutions that is integer solutions is seven

so again you can see that the partition problems the ah solutions in integers that is the diophantine equations are also basically problems in

combinatorics ah let us take another problem which is a permutation problem ah this is again from one of the je question papers three boys and two girls stand in a queue in how many ways they can be arranged

so that the number of boys ahead of every girl is at least one more than the number of girls ahead of her ah

so here the boys and girls they will be identifiable

so we can give some identification to them

so let wires be  $b_1, b_2, b_3$  and girls there are two girls  $g_1$  and  $g_2$

so basically we have to place them in a queue

so that means they are standing in a sequence here now let us look at the placements here which will satisfy this condition we can have following ah

so we have to keep track of the condition

so let us look at this suppose i consider  $g_1, g_2, b_1, b_2, b_3$  that means firstly two girls are standing and then three boys are standing

so now let us look at ahead of  $g_1$  you have one girl and three boys

so the condition that the number of boys ahead of every girl is at least one more than the number of girls ahead of her

so ahead of  $g_1$  you have one girl and three boys

so three is at least one more than one because three is two more than one similarly ahead of  $g_2$  the number of girls is zero and number of boys is three

so again this condition is satisfied

so basically what we are saying is that we put two girls first and then three boys how many such arrangements are there

so ah these two girls can be permuted in ah two factorial ways and these three boys can be permuted in three factorial ways

so we can permute two girls in two factorial ways and three boys in three factorial ways

so total number of such arrangements is ah  $2! \times 3!$  we have applied the multiplication principle

so this is six into two that is twelve now let us look at another arrangement in which you have

so this is one arrangement in the second arrangement what we do we have a girl first then a boy then a girl and then two boys

so let us look at this ahead of  $g_1$  you have one girl and three boys

so the condition that the number of boys ahead of every girl is at least one more than the number of girls i had offer this condition is satisfied similarly ahead of  $g_2$  we have ah two boys and no girl

so again this condition is satisfied again if you look at the arrangement of the boys and the girls

so among girls you can have permutation that means you can have g two here and g one here and b one b two b three also can be permuted in three factorial ways so again ah boys can be permuted in three factorial ways and girls can be permuted in two factorial ways

so total number of

so total number of permutations or arrangements is three factorial into two factorial that is twelve

so what we have done we have counted in which two girls occupy the first place and the three boys occupy the next three places second option we have taken in which ah there is a girl followed by a boy and then there is a girl and then followed by two boys let us look at the third possibility we can have a girl first and then you can have a boy then you can have another boy then you can have a girl and then you can have ah a boy again let us look at this ahead of this girl there is a boy then there is no girl ahead of it

so condition that the number of boys is at least one more than the number of girls ahead of that is satisfied and with respect to g one certainly the condition is satisfied because only one girl is ahead and all the three boys are ahead and again you have three factorial into two factorial that is twelve such arrangements ah another possibility that you can have is you can firstly have a boy then the two girls and then the two boys again let us look at the condition ahead of the first girl there is one girl and two boys ahead of the second girl there are two boys and no girls again the condition that the number of boys ahead of every girl is at least one more than the number of girls ahead of her is satisfied and again because of the permuting of the girls and the boys here you can have three factorial and two two factorial that is twelve such arrangements in which at the first place you have a boy two girls and then there are two boys the next one you can have why then there is a girl then there is a boy then there is a girl and then there is a boy again if you look at the conditions ahead of g two you have one y ahead of ah g one you have one girl and two boys

so again the conditions are satisfied and these can be permuted into three factorial into two factorial that is twelve such arrangements are there

so if you look at the total number of arrangements total number of such arrangements

so we have twelve into five that is sixty

so we have used here the addition principle as well as the multiplication principle for finding out the arrangements of ah three boys and two girls in a particular way ah let us consider a particular word ok consider e n d e a n o e l all right

so if you count there are nine letters here

so firstly we ask how many words can be formed by considering all letters of this word total number of permutations here

so here if you see the total number is nine

so you have nine letters out of which if you look at e e is repeated three times and if you look at n then n is repeated twice and then you have one d one ah a is there and then you have o and you have l these are appearing once

so if you look at that how many permutations the total arrangements is nine

so if you have three things which are identical and two things are identical

so you will have nine factorial divided by three factorial into two factorial of course one can simplify this ah there will be total thirty thousand two hundred forty such arrangements ah now suppose we fix that a particular word occurs there in how many words the segment say a particular segment n e and d e a occurs now out of this nine letters if you choose this five letters which have to necessarily appear then the total number of things that are there that will

be only four plus one five because these have to appear together

so ah here the number of objects is five that is e n d e a and then you have the remaining one e ah one o one l and one ah n

so the number of arrangements is five factorial that is one hundred twenty similarly let us see another restriction here ah the number of words is starting and ending with say letter e how many such letters will be there

so if we fix the first and the last then we are left with seven out of which n is appearing twice

so now we have seven letters remaining in which n occurs twice

so the number of arrangements will become seven factorial divided by two factorial that is two thousand five hundred and twenty in the same one let me put one more restriction in how many words letters a e o occur only in odd positions ok in order to ah look at this one

so there are three three e's ok and total odd positions they are one three five seven and nine

so in five places a o and three e's

so these are also now five they can be placed in how many ways five factorial divided by three factorial ways now in remaining four positions l d and two n's these are also four can be placed in four factorial divided by two factorial ways

so you have total number of such words that is equal to 5 factorial divided by three factorial into four factorial divided by two factorial that is two hundred forty that is there are two hundred forty words in which the letters a e o will appear in odd positions ah let us look at another variation of it the number of permutations in which none of the letters d l and n occur in the last five positions

so d l and two ns must appear in first four places that is fixed now the number of ways will be four factorial divided by two factorial here and remaining a o and three e's can be placed in last five positions in five factorial divided by three factorial is

so the total number again becomes four factorial divided by two factorial into five factorial divided by three factorial that is equal to two hundred forty

so the number of permutations of the letters of the given word in which d l and the two ends they occur in the first four positions and the remaining occur in the last five positions is again two hundred and forty ah let us take up another problem which has appeared in the joint entrance examination this is also from one of the je question papers in how many ways five wires and five girls can be placed in a queue

so that one all girls stand consecutively are exactly four girls stand consecutively ah

so now out of the five wires they can be anywhere but if the all the five girls are together then they can be treated as one unit

so therefore the total number of objects can be five plus one six

so they can be placed in six factorial ways however these five girls they can permute among themselves

so it will become five factorial

so if all the girls stand consecutively they can be treated as one unit

so total number of permutations of five wires plus one unit is six factorial however the five girls can be permuted in five factorial ways

so the total number of such arrangements that will be six factorial into five factorial

so of course that is a large number ah eighty six thousand four hundred now suppose i look at that out of this five exactly four girls are together and one girl is at some other place that means it is not in the same sequence let us see

in how many ways this can be done

so if four girls are together then they can be treated as a single unit four girls ah who stand consecutively can be treated as one unit ok ah

so five boys and these four girls can stand in six factorial ways and these ah four girls can also be permuted in four factorial ways further these four girls can be permuted in four factorial ways another thing is that out of the five girls we have chosen four girls two stand together and one girl is separate

so that can also be chosen in either  $5C1$  or you can say  $5C4$  ways ah the girl who stands separately can be chosen from five girls in  $5C1$  ways

so let us look at the arrangements now ah you can have one two three four five and six ah that is the four girls treated as one unit and five wires they are standing in six factorial ways and then there are other counting also

so let me just say that they are placed here

so for example ah this could be these four girls then this could be the next five wires or this could be four girls and this could be five boys or this could be that four girls and this could be five wires etcetera or this could be four girls and these could be five boys now this one girl which is remaining can stand here here here here here here or here that is there are total number of possibilities seven however since we have the put condition that exactly four girls stand consecutively

so if these are four girls then the remaining girl cannot be here or it cannot be here that means it has to be placed in the five places similarly wherever those four girls occur adjacent to that both the sides that girl cannot stand

so remaining only five possibilities are there for placing that last girl here

so the last girl can be placed in five ways as it cannot be placed adjacent to previous four girls

so now if we apply the multiplication principle the total number of arrangements

so that is  $6!$  into  $4!$  into  $5C1$  into five

so of course one can evaluate it it is actually four lakh thirty two thousand ah here you have seen that how the counting has been done we say exactly four girls stand consecutively

so they are treated as one unit now one girl is remaining

so we consider it separately and there are five wires

so five wires plus this unit of four girls that becomes six

so they can be placed in six factorial ways now these four girls can themselves permute in four factorial ways now another choice is coming because one girl is separated

so out of the five girls the number of ways of choosing that girl will be  $5C1$  now the placing of this girl has to be in such a way that it is not adjacent to the unit of four girls now that unit of four girls wherever it is placed

so there are actually seven places where one can put this last girl

so leaving aside two adjacent to the set of four girls you have only five available options

so that is the number of ways is five

so now the total number of ways we have applied the multiplication principle  $6!$  into  $4!$  into  $5C1$  into five that is four lakh thirty two thousand is the number of ways ah

so let us look at some more permutation combination problems

so a player and i named lisa chooses 13 cards from a deck of 52 cards in how many ways she can choose

so that she gets two kings and two queens that means the card which are

displaying king or queen

so now there are 52 cards out of the 52 cards you have four king cards and four queen cards and then remaining cards are forty four

so if you are putting this restriction that out of these thirteen cards which are selected two must be king

so that means they have to be chosen from these four

so that number will be  $4C2$  similarly we are also getting two queens

so again by multiplication principle this will be multiplied to this number  $4C2$  now this also places the condition that the remaining nine cards can be any of the cards from the forty four cards

so that is  $44C9$

so this is a simple combination problem because these are unordered arrangements here

so we have just counted straight away that there are four kings

so out of that two are chosen in  $4C2$  ways there are four queens out of that two are chosen in  $4C2$  ways and out of the remaining forty four cards we choose nine in  $44C9$  ways ah let us take another problem of combination in how many ways three balls can be chosen from a set of three red four blue and two green balls

so that all are of different colors are all are of the same color

so we are choosing three and if we say that all are of different colors that means we must choose one red one blue and one green ball now the number of ways of choosing that can be  $3C1$  for blue  $4C1$  and for green it is  $2C1$

so this number is straightforwardly twenty four now in the second part we are saying that all are of the same color now since we have chosen three and if they have to be of the same color then either they should be must all be red or all must be blue

so if all of them have to be red then this will become in  $3C3$  ways that is simply one r all of them can be blue which can be chosen in  $4C3$  ways

so that is one plus four that is five ways obviously you cannot have them to be green because only two green balls are there

so if you can see carefully in the first part we have applied the multiplication principle and in the second part we have applied the addition principle here by adding the two possibilities that is all of them can be red or all of them can be blue here ah there are more such permutation problems ah

so all permutations of the word last are arranged according to english dictionary ordering ah what is english dictionary ordering that is ah if we look at the letters here a l s and t

so the first word will become a l s t and then ah next one will be that starting from a and then you can say alts and like that that means the exact dictionary ordering has to be followed what is the position of the word say salt in this ordering

so that means if we write all of them together that is basically there are ah twenty four such words here because all of them are distinct

so four factorial in these twenty four words what is the position of the word salt

so firstly we consider the words which are starting with a how many such words are there the number of words is starting with a

so that first word is a and in the remaining you have l s t and they can be permuted in three factorial ways

so total number of words will become six now the number of words is starting with l

so if we keep l in the first place and then a and s t can be permuted again in three factorial that is six ways then the next one ah since we have according to the dictionary ordering you have a l s and t

so therefore the next one will become s now if you look at the dictionary ordering a will be the next one coming here then l and then t

so that means the position of salt will be 13th in this list

so that is thirteenth position here ah in the next lectures i will be considering some more ah principles of counting ah some occupancy problems and then we will solve few more exercises of this nature you