

in the last two lectures i have introduced the basic principles of counting
so you have the addition principle you have the multiplication principle and
then you have the concept of number of arrangements and in the concept of number
of arrangements you can have permutations in which the ordering is important and
if ordering is not taken into account then we call it combinations

so we introduce the coefficients $n P k$ that was n factorial divided by n minus
 k factorial and we introduced $n C k$ that is a number of k unordered samples from
 n

so that is n factorial divided by k factorial n minus k factorial

so now we will be discussing various properties of ah these coefficients $n P k$
and $n C k$

so $n P k$ is the number of k ah k items taken together

so k ordered samples from n ah

so we had discussed one property that is $n C k$ is same as $n C n$ minus k let us
look at further properties

so one property is that r into $n C r$ that is same as n into n minus $1 C r$ minus
 1 where of course r is greater than or equal to 1 and r is less than or equal to
 n let us look at the proof of this property

so let us consider the left hand side that is r into $n C r$ ah you note here
that i am using another notation here

so $n C k$ i can also write as $n C k$ like this

so both the notations can be used they are equivalent

so r into $n C r$ that is r into n factorial divided by r factorial into n minus
 r factorial ah

so now let us look at this we can adjust the terms in the numerator i write n
factorial and this r and here in the denominator there is r factorial which is
nothing but r into r minus one factorial

so the r in the numerator and denominator can be cancelled and you have r minus
 1 factorial then this next term i can write as n minus 1 minus r minus 1
factorial

so this numerator this n factorial i can write as n into n minus 1 factorial
that means i separate out n here

so this is now equal to n and this term if you look at n minus 1 factorial
divided by r minus one factorial and n minus one minus r minus one factorial

so this is nothing but n minus one choose r minus one

so that is the right hand side here ah

so the proof of this is simple let us look at the ah physical understanding of
this

so $n C r$ denotes the number of unordered samples of r items taken from n items

so if we multiply this by r what does it mean it means that r times such thing
is to be considered now what we say that it is same as choosing r minus one
things unordered samples from n minus one things and if you multiplied by n

so here it is n minus one

so if you do it from n such samples then that number is same as here that if
you choose r things from n and you do r such things

so that is a same ah let us look at similar other properties n minus r into $n C r$
 r is same as n into n minus one $C r$

so let us look at this left hand side n minus r into $n C r$

so that is n factorial divided by r factorial into n minus r factorial ah here
you can see this n minus r and n minus r factorial here the first term will
cancel out

so you will get n factorial divided by r factorial n minus r minus one
factorial what we do we adjust the term here in the numerator i separate out n
and write it as n minus one factorial and then you have r factorial n minus one

minus r factorial

so this is n minus r minus one i write as n minus one minus r then this quantity you can see is nothing but n minus 1 choose r

so this is the quantity i wanted to exhibit ah

so this property basically is a restatement of the previous property because nCr is same as $nCn - r$

so this property and this property are of the similar nature let us look at another property $rCn - r$ is equal to $n - r + 1Cn - r - 1$ one of course in all these cases you have that r lies between one and n here

so if you look at the left hand side that is r into nCr that is r into n factorial divided by r factorial $n - r$ factorial ah once again you can see this r and r factorial here you can cancel the first term here

so you get $r - 1$ factorial and $n - r$ factorial here i multiply by $n - r + 1$ in the numerator and denominator

so i will get $n - r + 1$ factorial now this term is nothing but $nCr - 1$

so that is the term that you wanted to show here ah we have some properties of the permutation also

so let me give you of this $n + 1Pr$ that is equal to $nPr + r$ into $nPr - 1$ ah let us firstly look at the proof of this

so if you look at the right hand side that is nPr that is n factorial divided by $n - r$ factorial the second term is r into n factorial divided by $n - r + 1$ factorial

so that is equal to now you can take out n factorial from here and here i multiply and divide by $n - r + 1$

so $n - r + 1$ divided by $n - r + 1$ factorial plus r divided by $n - r + 1$ factorial

so in the numerator if you see $n - r + 1 + r$

so it becomes simply $n + 1$

so that is $n + 1$ factorial because $n + 1$ into n factorial divided by $n - r + 1$ factorial that is nothing but $n + 1Pr$ which is the left hand side ah let us look at the interpretation here on the left hand side this is the r items chosen from $n + 1$ and we are looking at basically ordered arrangements of r things from $n + 1$ things

so what this result says that it is the same as that if you choose r things out of n things

so on and order them that is r ordered arrangements out of n things and $r - 1$ things from n things taken r at a r times

so this number will be the same as number of ordered r sets out of $n + 1$ cells ah let us look at a similar such property $n + 1Pr$ is same as r factorial plus r into $nPr - 1$ plus $n - 1Pr - 1$ and

so on plus $rPr - 1$

so if we take the last term here

so we have right hand side and i combine one by one the terms

so using the result of exercise four that is this property here what we are saying is that r times $nPr - 1$ plus nPr becomes $n + 1Pr$

so if you look at this last term here

so here it is r times multiplied by $rPr - 1$

so here if you add the first term that is rPr

so rPr plus r into

so let us look at this r factorial and this term i take $rPr - 1$

so it is nothing but rPr plus $rPr - 1$ into r

so if you look at this thing

so here in place of n i am putting r

so this should become equal to $r + 1 P_r$ okay now this term has come and what will be the next term here the next term here is ah

so now we have $r + 1 P_r$ and here the term will be $r + 1 P_{r-1}$
so again it is the same thing as this exercise by putting n is equal to $r + 1$

so this will become $r + 2 P_r$ now again you add this here and the next term here that is $r + 2 P_{r-1}$.

so like that you continue the last term will give me $n P_r + r + 1 P_{r-1}$ that is $n + 1 P_r$

so let me show it here in order to prove this property i am making use of the previous property here now this property is valid for one less than or equal to r less than or equal to n

so i will choose different values of n

so if i put n is equal to r and i look at this term r factorial is nothing but $r P_r$ and the last term here that is $r + 1 P_{r-1}$

so this by this property it becomes $r + 1 P_r$ now this term again i combine with the second term here that is $r + 1 P_{r-1}$

so that is the term given here $r + 1 P_r + r + 1 P_{r-1}$

so by this property again by choosing n is equal to $r + 1$ i will get it as $r + 2 P_r$ now again i combine it with the next term here and

so on ultimately i will get the term $n P_r + r + 1 P_{r-1}$.

so if i put from here this becomes $n + 1 P_r$

so this property is established here ah another property of the permutations is $n - r + 1 P_r$ that is equal to $n - r + 1 P_r$

so if i consider the left hand side that is $n - r + 1 P_r$ divided by $n - r + 1 P_r$ ah

so you can cancel out $n - r + 1$ in the numerator and denominator and you get the term $n - r + 1 P_r$ which again i write as $n - r + 1 P_r$ divided by $n - r + 1 P_r$ that is nothing but $n - r + 1 P_r$ ah

similarly we can have like $n - r + 1 P_r$ is equal to $n - r + 1 P_r$ let us look at the proof of this the right hand side is $n - r + 1 P_r$ divided by $n - r + 1 P_r$ now here the first term is $n - r + 1$ that cancels out

so you get $n - r + 1 P_r$ divided by $n - r + 1 P_r$ that is $n - r + 1 P_r$ ah

so again ah the interpretation of this is here we are looking at the r ordered arrangements from n things taken at a time and that is the same as $n - r + 1$ such things

so that is the multiplication here giving the thing ah the next property is also similar in nature $n P_r$ is same as $n - r + 1 P_r$.

so the right hand side is $n - r + 1 P_r$ that is $n - r + 1 P_r$ divided by $n - r + 1 P_r$ that is same as $n - r + 1 P_r$ we can use this concept of permutation and combination to prove some results in number theory also i will give one example here

so if i consider say product of n consecutive integers then it is divisible by n factorial

so let us consider say n consecutive numbers

so let us consider say $r + 1$ $r + 2$ and

so on $r + n$

so these are n consecutive integers

so here r and n are positive integers ah we have to prove that this is divisible by n factorial

so here i will give a proof in which we will make use of the permutation and combination

so this we can write as say if we consider these terms it is n plus r n plus r minus 1 and

so on

so here if i multiply by r factorial

so i will get r factorial into this term that is p divided by r factorial

so if i consider r factorial n to this then this is nothing but n plus r factorial divided by r factorial and now we can consider this as further n factorial multiplied by n factorial

so you look at this this is nothing but n plus r choose r into n factorial

so what we are getting p divided by n factorial that is n plus r C r now this is the number of

so let me go back to the definition of ah combination that i gave in the previous lecture

so if we use this exact definition n C k denotes the total number of k combinations from a set of n distinct items

so these are unordered arrangements

so this is a number okay

so since this is a number

so that means p divided by n factorial is a number that means p is divisible by n number of unordered r combinations out of n plus r distinct objects

so p is divisible by n factorial

so let us solve some problems ah for example we want to find the sum s is equal to say 1 into one factorial plus two into two factorial plus and

so on n into n factorial

so what we do we write this s as this one i can write as two minus one into one factorial plus this two we can write as three minus one into two factorial ah the next term will be three into three factorial

so three we can write as four minus one into three factorial and

so on finally n plus 1 minus 1 into n factorial

so this is nothing but 2 into 1 factorial which is nothing but 2 factorial minus 1 factorial plus 3 into 2 factorial minus 1 into 2 factorial plus 4 into 3 factorial minus 3 factorial and

so on finally n plus 1 into n factorial minus n factorial

so this i since each of these factorials is multiplied by the next consecutive number this is becoming two factorial minus one factorial now this term again becomes three factorial minus two factorial plus four factorial minus three factorial plus and

so on n plus one factorial minus n factorial now you notice the terms here this has become a telescopic sum that is the first term here is the same as minus second term like two factorial minus two factorial similarly here you will have three factorial in the next one you have minus three factorial

so these terms cancel out

so ultimately all the terms will get cancelled out and we will be left with n plus one factorial that is the last term minus the second term here that is one

so this sum of this series is nothing but n plus one factorial minus one ah

there are six boys and five girls waiting to be seated on 11 seats in a health spa ok ah two particular boys are named as say ramesh and giri and say one girl one particular now we will have arrangement

so find the number of ways of seating all boys and girls find number of ways of seating

so that say ramesh and giri they are adjacent that means they sit side by side and thirdly find the number of ways of seating

so that ruby is in the middle seat ramesh is on a seat on left side of ruby and giri on right but not necessarily adjacent

so let us look at the number of ways here ah there are 11 children here 6 boys and 5 girls and there are 11 seats

so if we have to seat all of them then in how many ways we can do that

so in the first part it is exactly choosing eleven things and then arranging them now these boys and girls will be distinct because they are identifiable therefore the number of arrangements is nothing but the number of ordered arrangements for example if i choose 2 out of them say ramesh and giri

so firstly ramesh is sitting then giri are firstly girish and then ramesh

so in these two orders if they are

so already two factorial will come

so now it is eleven

so the total number of ways when there are no restrictions will become 11 factorials

so the total number of ways of seating of ramesh and giri that is 11 factorial you can argue in a different way also the first person can be seated in eleven ways the second person can be seated in ten ways the third person can be seated in nine ways and

so on

so eleven into ten into nine and

so on up to three two one that is again eleven factorial

so we can say in either way $11P_{11}$ or we can say eleven factorial both will give the same answer ok ah in the second one we are putting the restriction that ah ramesh and giri sit on the adjacent seats

so if they sit together then we can treat them as one entity if ramesh and giri are in adjacent seats then we can treat them as one entity

so now there are ah

so now the arrangements of these ten things ten factorial arrangements because wherever this ramesh and giri have to appear they have to appear together but certainly they can themselves interchange their positions however they can interchange their places

so two into ten factorial that is the total number of phase let me repeat this here in the second part i am choosing ramesh and giri to always be sitting together

so if they have to always be sitting together then there are nine other children

so nine plus this ramesh and giri i consider them as one entity

so it is becoming now ten things now these ten people have to be arranged now the number of ordered arrangements will be $10P_{10}$ that is 10 factorial among ramesh and giri again they can interchange their positions

so that is 2 times

so now if you apply the multiplication principle it becomes 2 into 10 factorial

so that is the total number of ways of seating these 6 boys and 5 girls such that 2 of them are always together ah now to look at the third part here in the third part we are putting some restriction here that ruby is in the middle seat and ramesh is on a seat left side of ruby and giri is on a seat which is to the right side

so let us look at the positioning here

so we just make some sort of diagram just to illustrate this here

so you have

so you have we can just say names seats one two three four five six seven eight

nine ah ten and eleven

so there are total eleven seats in the middle seat ruby is there

so her seat is fixed here now this ramesh can be on any of these seats and similarly giri can be on any of these seats here

so what is the total number of possibilities

so ruby occupies middle seat that is seat number six

so there is only one possibility here now ramesh can be seated in five c one that is five ways because he can be seated in any of these five places and similarly giri can be seated again in five c one that is five ways now three persons we have seated out of eleven

so there are eight persons left out

so they can be now how many ordered arrangements of them will be there that is $8 P 8$ that is 8 factorial

so remaining eight children can be seated in eight remaining seats in eight factorial ways

so now we apply the multiplication principle by multiplication principle the total number of seating plans is five c one that is five into five into eight factorial

so of course one can ah evaluate this as you can understand this value of eight factorial is large and then you have to again multiply it by twenty five ah let me put one more problem in the same one in the above problem in how many ways boys and girls can sit in alternate seats that means one y then girl then boy then girl like that

so if we put like that then again let us look at this ah let me make this arrangement again um eleven places are there now there are six boys and five girls

so if we start with a boy then a girl will be coming here then boy then girl then boy then girl then boy then girl then boy then girl and then boy

so immediately you can see that in this arrangement the boy has to be in the odd number of places that is one three five seven nine and eleven ok and then in the middle five places which are even two four six eight and ten here the girls can be seated now if you change the order for example if you start with a girl then you can immediately make out that one boy will be left out because if you start say from here then one seat one boy will be left out and you will have to put here then they will not be alternating

so since the number of boys is exactly one more than the number of girls the exact number of plan seating plan will be only in like this

so what you have to see that how many such arrangements are possible

so we can see here that six boys can occupy six odd numbered seats now that is in six factorial ways and in remaining five even numbered seats five girls can be seated in five factorial ways

so again by multiplication principle the total number of ways in which boys and girls sit in alternate seats is six factorial into five factorial

so here you can see that how the counting of the permutations or you can say ordered arrangements is there ah now let me take up one or two problems for combinations in which unordered arrangements are there

so ah in a school 11 of 25 teachers are in favor of say value education eight are against and three are neutral ok

so whether the course on value education should be introduced or not

so 11 teachers favor that eight teachers oppose that and three have no opinion they are neutral in how many ways five teachers can be chosen

so that they are in favor of value education or secondly they have the same opinion or say third two are in favor two are against and one is neutral

so let us look at the solution here now here you can see that it is unordered

arrangements because if we are choosing say five teachers in which order we have chosen does not make any difference because they are not being seated or something like that we are just choosing them

so it is a set of the number of unordered subsets of size 5 out of 25 under certain restrictions

so in the first case i am asking that they are all in favor of value education that means the 5 teachers which are chosen they have to be from this eleven only

so it is nothing but eleven choose five in the second case they have the same opinion now if they have to have the same opinion then maybe they are all in favor or they are all against

so eight c five and ah i think i made the calculation wrongly here this has to be six c five are neutral or it could be that all of them are neutral

so six c five ok

so what we have calculated in how many ways five teachers can be chosen out of twenty five such that they have the same opinion

so either all of them have to be in favor of value education and that number is nothing but eleven choose five are they are all against it

so that number will be eight c five or all of them are neutral then that number is six c five

so here what we have done we have applied the addition principle

so this total number of codes can be easily calculated now the next one is that we choose five in such a way that two are in favor two are against and one is neutral

so if two are in favor that can be chosen in eleven c two ways two are against that can be chosen in eight c two ways and one is neutral can be chosen in six c one ways and then you have to apply the multiplication principle

so it becomes eleven c two into eight c two into six c one of course these numbers can be easily calculated let us consider another problem on the combinations in a lottery eight numbers are chosen from 1 to 99 as winning numbers if all numbers match the person who is betting

so let me say just we put some arbitrary name say john vince the first price

so he chooses eight numbers and if all the eight numbers match with the numbers are given in the lottery then he gets the first prize ah if seven numbers match then john gets second price and if six numbers match john gets the third price in how many ways john can choose numbers

so that he gets some price now if he has to get the first price then the all possibilities have to be exactly the same as the numbers which are given by the lottery that means suppose number 3 is chosen then he has to get 3 suppose number 7 has to be chosen then 7 has to be there if it say 13 45 or whatever numbers are there then that set is identified as the numbers allocated for the lottery

so the number of ways of getting the first price is exactly one ah the number of ways of getting the second price now in the second price he should be able to get seven numbers out of those eight and one number can be some other number

so now that has to be eight c seven out of 99 now remaining 91 numbers are there

so he will get one number out of the remaining 91 numbers and we apply the multiplication principle here because total numbers chosen is eight

so this seven numbers have to be the one which are out of those eight numbers and any one number can be different of course

so you can evaluate it is seven hundred twenty eight then again let us look at the number of ways of getting the third price

so that is 8 c 6 91 c 2 because the third price is if the 6 numbers match

so these 6 numbers have to be from the eight numbers which are given in the

lottery then the remaining two numbers can be any other from ninety one numbers
so this of course again can be evaluated that is twenty eight into four
thousand ninety five that is eleven thousand four six six zeros ah
so now you apply the addition principle the total number of ways of winning the
price

so that is plus seven twenty eight plus one

so that is one one five three eight nine one lakh fifteen thousand three
hundred eighty nine is the total number of ways in which he can actually win the
ah win a prize of course if you think that ah this is a huge number then only
you have to compare it with the total number of possibilities ah then the total
number of possibilities will be 99×10 which will be actually a very very huge
number ah another counting exercise in which you can use permutations find the
number of integers between three thousand and six thousand in which and each
digit is not repeated that means how many numbers are between 3000 to 6000 in
which a digit is not repeated that means three thousand itself is not considered
so for example three thousand one cannot be considered because zero is repeated
similarly suppose i say four thousand one hundred twenty two then that number is
also not counted

so let us see

so the first digit can be 3 4 r 5 because the number has to be between thousand
and six thousand

so there are three cases now other three digits can be chosen from remaining
nine digits nine digits means see you are considering zero one two nine

so ten digits are there now one digit has been taken out

so remaining nine digits are there from there you have to choose three but
distinct and then it becomes nothing but the number of permutations taking three
at a time from nine

so nine p three

so the total number of of such digits is $3 \times {}^9P_3$ that is $3 \times 9 \times 8 \times 7$
factorial divided by $n - k$ that is six factorial that is one thousand five
hundred twelve

so out of basically three thousand numbers one thousand five hundred twelve
numbers are the ones in which no digit is repeated ah now let me just find
another problem here say in the above problem find the number of even digits

so let us look at this the first digit the first digit number of even numbers
basically i should say

so the first digit ah if the first digit is say 4 okay then the last digit can
be from 0 two six eight that is four ways now second and third digit can be
chosen from the remaining eight digits because two digits have been taken first
digit is four and the second digit has been chosen to be an even number out of
these four

so two remaining digits can be chosen in 8×7 ways all right

so the number has become $4 \times 8 \times 7$ that is 4×56
factorial divided by six factorial that is two hundred twenty four let us also
look at the first digit can be three or five if the first digit is three r five
then that is two ways ok then the last digit can be from 0 to 4 six eight that
is five ways and again remaining two digits can be chosen in 8×7 that is
eight factorial divided by six factorial base

so what is the total number of ways two into five into fifty six that is five
hundred sixty

so now you just add these two

so by addition principle the total number of even integers between 3000 and
6000

so that digits are not repeated is two two four plus five six zero that is

seven hundred eighty four ah let me repeat this statement here total number of digits integers between 3000 to 6000 where the digits are not repeated is one thousand five hundred twelve out of them how many are there where even numbers are there

so that is out of that seven hundred eighty four are the ones where numbers are even ah in the next lecture i will continue ah various other applications of this problems of permutations and combinations ah basically in the permutations number of ordered arrangements and in the combinations the number of unordered arrangements

so there can be various type of problems where these things are applicable so we will be discussing some more problems in the next lecture you