so firstly let me start with the few more counting methods
so one of them is the
so called the injection principle ok
so these principles are based on counting the number of ah functions from ah one finite set to another finite set ah depending upon the nature of the function that means it can be a one to one function or it can be an onto function etcetera
so depending upon that you will have the number
so if those numbers are matching these are used for doing various counting problems
so let me firstly talk about the injection principle
so let a and b be finite sets if there exists a one to one mapping f from a to b then the number of elements of a is less than or equal to the number of elements of b ah
so this is something very simple to understand for example i just make a diagrammatic representation i have certain number of elements here and
so each one of them is it is a one to one function
so so naturally you can see that each element is mapped to a member here
so if there are some members left out then the cardinality of b is always more than or equal to the cardinality of the set a
so this is only less than or equal to
so if we put one additional restriction on this then you have the direct counting
so that is called the bijection principle or we call it ah bp
so let a and b be finite sets if there exists a bijection f from a to b that means ah one to one and onto mapping then the cardinality of a is same as the cardinality of b ah we can also talk about ah many times there are problems in which we are asked to find out the number of divisors of a given natural number
so here what we do we can actually consider ah similar thing that is by bijection method ah let us consider this
so applications of fundamental theorem of arithmetic ah usually we call it f t a in chart for finding the number of divisors of a natural number what is the fundamental theorem of arithmetic i think all the students are familiar with this that is every natural number n greater than or equal to two can be factorized as n is equal to p 1 to the power m 1 p 2 to the power m 2 p k to the power m k for some distinct primes p one p two p k and some natural numbers m one m two m k such a factorization is unique if we disregard the order of primes that means in place of p 1 to the power m 1 into p to the power m 2 etcetera if we interchange the order firstly we put p 2 to the power m 2 then we write p 1 to the power m 1 then it does not make any difference then this factorization is considered to be unique ah this is one of the fundamental results in number theory in fact it was known at the time of euclid who proved it for one elementary case and in the full form it was proved by carl frederick gass in eighteen hundred and one ah now let us see this can be used for finding out the all the divisors
so i will show you the application of the bijection principle here find the number of divisors of say seventy two
so seventy two can be written as two cube into three square
so by factorization theorem this is the ah unique representation of this
so if x is a divisor of seventy two then x can be written in the form x is equal to say 2 to the power a into 3 to the power b where a belongs to the set 0 1 2 3 and b belongs to the set 0 1 and 2.

so so now let us consider let a be the set of divisors of 72 and b be the set of this tuples a b where a can take values 0 1 2 3 and b can take values 0 1 2.
then naturally you can see that there is bijection from a to b ah in fact we can display this full thing in fact we have function you can write like this see this f if i choose say 0 0 that means a zero b zero you get exactly one ah
so i have to say it is like this f one that is the divisor one is corresponding to a is equal to zero b is equal to zero if we consider the divisor two then that is corresponding to a is equal to one and b is equal to zero similarly if i consider divisor say three then that is corresponding to a is equal to zero b is equal to one if i consider divisor four then that is corresponding to a is equal to two b is equal to zero if i consider divisor six then that is corresponding to a is equal to one b is equal to one then if we consider the divisor eight then that is corresponding to a is equal to three b is equal to zero if we consider nine then that is corresponding to a is equal to zero b is equal to two if we consider twelve then that is corresponding to a is equal to two and b is equal to one if we consider say eighteen then that is corresponding to a is equal to one and b is equal to two if we consider twenty four then that is corresponding to a is equal to three b is equal to one thirty six if we consider then that is corresponding to two two that is four into nine and finally f seventy two is corresponding to three two
so if you look at 1 2 3 4 5 6 7 8 9 10 11 12 all the 12 elements of the set a are corresponding to total number of twelve elements of the set b
so it is a one one on two map that is a bijection
so by the bijection principle
so by the bijection principle the number of elements of a is same as the number of elements of b that is twelve there are total twelve divisors of the number seventy two ah
so now let me just give some ah simple applications to some slightly larger numbers
so we look at find the number of divisors of say twelve thousand six hundred thirty one thousand seven hundred fifty two are fifty five thousand one hundred twenty five
so let us look at this twelve thousand six hundred can be factorized as two cube into three square into five square into seven to the power one
so any divisor is of the form 2 to the power a 3 to the power b 5 to the power c and 7 to the power d where now you see a b c d where they belong to a can be any number zero one two three b can be any number zero one two c can be any number zero one two and d can be any number zero one
so so mapping f defined by f a f of x is equal to a b c d then this is a bijection from a that is the set of divisors of twelve thousand six hundred to the set b which is nothing but the collection of four tuples where ah a is between zero to three b is between zero to two c is between zero to two and d is between zero to one they are integers now how many elements are here cardinality of b is ah here you have four elements here you have three elements here you have three elements and here you have two elements
so this is nothing but exactly seventy two
so the number of divisors number of divisors of twelve thousand six hundred is seventy two ah similarly let us consider say three thirty one thousand seven hundred fifty two ah this number is two cube into three to the power four into seven square
so if we consider x as two to the power a into three to the power b into seven to the power c then a b c will belong to zero less than or equal to b less than or equal to four zero less than or equal to c less than or equal to two
so cardinality of b is four into five into three that is sixty

so that is the number of divisors of this in a similar way if i consider fifty five thousand one hundred twenty five then that can be expressed as square into five cube into seven square

so the cardinality of b in this case will become three into four into three that is thirty six

so in general then we can state the following theorem as an application of this fundamental theorem of ah arithmetic and ah the bijection principle we have the following result on the number of divisors of a given natural number as an application of f t a and the bijection principle now i stated in the form of a theorem if n can be expressed in the form p 1 to the power m 1 p 2 to the power m 2 p 3 to the power m 3 and

so on p k to the power m k where n is greater than or equal to two where p one p two p k are distinct primes and m one m two m k are natural numbers then the number of divisors of n is given by m one plus one into m two plus one and

so on up to m k plus one ah

so this was one application of the bijection principle which is useful in counting the number of ah divisors of a natural number i further look at some more counting methodologies ah

so they are called occupancy principles that is putting a number of objects into a number of cells or putting a number of balls in a number of boxes

so this has wide variety of applications when we are ah seating people when we are arranging ah some sort of cues etcetera

so ah in general they are called occupancy problems

so let me look at this the number of ways of distributing are identical balls into n distinct boxes n r plus n minus ones choose n minus one which is same as r plus n minus one choose r ah

so let me specify the problem

so as i mentioned you can have r identical objects

so which we can call our identical balls and they have to be placed in r distinct cells are n distinct boxes etcetera

so this number is given by r plus n minus 1 choose n minus 1 let me give a proof of this let us represent r balls by r stars

so this is just a way of representing sometimes we write it as r apples and in between we are placing ah oranges and

so on

so i am just putting the simplest notation stars ok

so this r balls they are identical

so they can be treated as this thing and indicate this n boxes by n spaces between n plus one vertical bars let me explain it

so suppose i have two boxes

so if i put three ah three vertical bars like this then this acts as a box this acts as a boss similarly if i have say three boxes then i make four bars

so so now what is happening is that the stars are in between because we are placing this r stars that is r identical balls into these boxes

so it is something like this you have r stars here and you have n plus one vertical bars again out of this vertical bars this there has to be one start and one end

so in between you can have n minus one vertical bars and r star

so that is r plus n minus one out of that n minus one things have to be chosen or you can say r things have to be chosen both are identically true

so it becomes r plus n minus one choose r r r plus n minus one c n minus one

so so this arrangement has vertical bars at the beginning and at the end remaining n minus 1 bars and r stars can be arranged in n plus r minus 1 choose r r n plus r minus 1 choose n minus 1 vs ah now in this arrangement there can be

possibility that ah some box is empty now suppose we put some additional restriction that no box is empty then what will happen to the number of possibilities let us look at that

so that is a variation of this occupancy problem the number of ways of distributing are identical balls in n distinct boxes

so that no box remains empty

so that is r minus one choose n minus one which is also r minus 1 choose r minus n here r is greater than or equal to n let me prove this statement also if there are no empty boxes then no two vertical bars can be adjacent for example in this i have made these two bars adjacent and there is no star in between

so this possibility will not be there

so the r stars have r minus 1 spaces of which n minus 1 have to be occupied by vertical bars that is how many vertical bars that is n minus one

so this can be done in r minus one choose n minus one that is r minus one choose r minus n ways ah

so i have given solution to two occupancy problems one is ah the number of ways of distributing are identical balls in n distinct boxes that is r plus n minus one choose n minus one and if we put the additional restriction that no box is empty then this number becomes r minus one choose n minus one

so let us look at an application of this 11 persons including three specific persons say p q r are to be seated on eleven seats

so that p q r do not occupy adjacent seeds in how many ways this can be done

so out of this eleven persons three persons are specific on which we are putting some restriction

so what we do we firstly place eight persons ok

so first we arrange eight persons in eight factorial ways that is we have excluded excluding p q and r

so you can see like this ah person one two three four five six seven eight now how many places are left out we have now seven places in between and on the side also we have two places

so there are total nine places in which we can put this p q and r ok

so this can be done in how many ways in nine remaining places three seats can be placed in nine p three that is nine into eight into seven ways

so the total number of possibilities becomes nine into eight into seven into eight factorial which is of course a large number ah six distinct symbols are transmitted through a communication channel a total of 18 blanks are to be inserted between the symbols with at least two blanks between every pair of symbols in how many ways symbols and blanks can be arranged let me just repeat the problem again there are six different symbols which have to be transmitted through a ah communication channel ah now in between the symbols we have to insert eighteen blanks such that between every pair of symbols at least two blanks are there then in how many ways this symbols and blanks arrangements can be done

so let the six symbols b say s one s two

so s one s two s three s four and s five ah sorry and s six ok now we have this spaces here ok this blanks have to be inserted necessarily between the symbols only

so between the symbols you have five places

so this can be considered as a problem of placing 18 identical balls that is the blanks we can treat them as the identical balls into five boxes now first thing is that it is written that between every pair at least two blanks have to be there

so firstly we put two two blanks there

so first we choose ah two blanks each

so from eighteen identical things if we choose ten identical things then we are left with
so since all of them are identical this choosing has no meaning basically it means one way only in one way now remaining how many are there eight blanks are left out now remaining eight blanks can be placed in five boxes in eight plus five minus one c five minus one that is 12 c 4 ways
so you can see this problem we have reframed as a problem of placing identical balls into distinct boxes plus we had some additional condition here that between every pair of symbols there are two blanks
so that part we took care by separately putting them
so 10 blanks were selected and put there since they are all identical then the number of ways of choosing that is one only now the remaining becomes r plus n minus one c r minus one and we have ah done that thing here find the number of non negative integer solutions to x one plus x two plus x three plus x four is equal to ten now this can be considered as the problem of distributing ten ones there are ten one
so x one can be zero ah because we are putting non negative integers
so x one can be zero one to ten x two can be zero one to ten and
so on
so basically it is a problem of distributing once
so there are ten distinct one identical ones which are to be placed into four distinct boxes
so this is same as distributing ten identical balls in four distinct boxes
so that is n plus r minus 1 c n minus 1 that is 13 c 3.

so we can actually extend this result to writing down an equation in n unknowns
so consider the equation x 1 plus x 2 plus x n is equal to r where r is a non negative integer let me call this equation number one then the number of non negative integer solutions to equation number one that will be r plus n minus one c r that is r plus n minus one c n minus one because this is same as the problem of distributing r identical things into n boxes if we put the restriction that x i's are greater than or equal to one that means non zero the number of positive integer solutions to 1 that is r minus 1 c n minus 1 r r minus 1 c r minus n ten letters are to be selected from five vowels repetitions allowed ok five vowels are a i e o u and we have to select 10 out of that that means all of these numbers can be repeated
so first thing is in how many ways this can be done second is what is the number if each vowel has to be selected at least once
so here if you see the solution here ten plus five minus one c ten that is same as fourteen c ten that is thousand one in the second case it is ten minus one c ten minus five that is nine c five find the number of ways of distributing are identical objects into n distinct boxes such that box one can hold at most one object let us give the argument for this here
so box one can have either no object or one object
so if box one has no object then the problem is to place r identical things into n minus one distinct boxes then we can distribute r identical objects in n minus one boxes in r plus n minus two c r ways and if we place if box one has one object then we can distribute r minus one identical objects in n minus one boxes in r plus n minus three c r minus one ways
so by addition principle the total number of ways is r plus n minus two c r plus r plus n minus three c r minus earlier we have considered the choosing of or you can say arranging of distinct items out of a list of distinct items in a particular way
so either unordered ways are ordered less that gave rise to the concept of ah

permutation and combination now this can also be considered as the distribution of balls problem into the boxes if we consider distinct balls and distinct boxes etcetera

so let me give this concepts also ah

so we now we consider distribution of distinct objects into distinct boxes the number of ways of distributing are distinct objects into n distinct boxes such that each box can hold at most one object

so that will be n p r let me give the proof of this here

so here you can see the difference from the previous one that earlier we had taken identical objects now we are taking distinct objects boxes were distinct earlier also now are there same

so this is actually nothing but the ordered arrangement let me give a formal proof of this

so you have say n distinct boxes now you are saying that you will put r objects into this

so this is same thing as like because each box can have at most one object

so here r is less than or equal to n here

so you can have say here here etcetera what wherever you put it is nothing but one arrangement of r things out of n

so this can be simply considered as number of are ordered arrangements of n distinct items that is nothing but n p r basically from n

so basically what you are doing is that you are choosing r boxes here out of n and here you are also saying that the number is important that means in which order we have placed them here ah we can give argument like this also one can argue in the following way also one can place the first object into any n boxes then second object into n minus one boxes and

so on

so rth object into n minus r plus one way boxes

so the total number of arrangements is nothing but by multiplication principle n into n minus one and

so on up to n minus r plus one that is nothing but n factorial divided by n minus r factorial that is n p r ah the number of ways of distributing are distinct objects into n distinct boxes such that any box can hold any number of objects n to the power r

so here you can see the argument is little bit different the first object can be placed into any of the n boxes the second object can be placed into any of the other n boxes and

so on

so each of our objects can be placed in n possible that is n into n into n r times that is n to the power r ways another way of counting is the number of ways of distributing r distinct objects into n distinct boxes such that the ordering of objects matters in each box

so then that is equal to n plus r minus 1 c r into r factorial that is n into n plus 1 and

so on up to n plus r minus 1.

so to resolve this problem let me first assume that let us first assume that objects are identical if that is

so then the number of the number of ways of placing these in n boxes is n plus r minus 1 c n minus 1 that is n plus r minus 1 factorial divided by n minus 1 factorial r factorial now if the objects are distinct and the ordering matters then the total number of orderings of r things is r factorial if the objects are distinct and the ordering matters then the total number of orderings of r objects is r factorial

so the total number of ways is n plus r minus 1 factorial divided by n minus 1 factorial r factorial into r factorial that is n plus r minus one factorial divided by n minus one factorial that is ah of course if you simplify this it becomes n plus r minus one two n plus r minus two and

so on up to which we can also write as n plus r minus one p r ah let me take up one another occupancy problem suppose there are n one identical objects of type one and two identical objects of type two and

so on and k identical objects of type k where n is n 1 plus n 2 plus n k that is the total number of n objects are there out of which n 1 are identical n two are identical and

so on they are all of ah different type then the number of arrangements of these n objects in a row is n c n one n minus n 1 c n 2 and

so on n minus n 1 minus n k minus 1 c n k which is same as n factorial divided by n 1 factorial n two factorial n k factorial

so this is actually same as arrangement of n objects in which n one objects are alike and two are alike and

so on

so this is same as n factorial divided by n one factorial and two factorial n k factorial

so this ah occupancy problems they have various applications we have already seen some problems which are ah solved using this arrangement that is you have distribution of things in two different cells ah arrangements by taking certain things in a particular order and

so on

so there are

so many problems that can be solved using these concepts

so right now i conclude my discussion on the permutations combinations occupancy problems and certain principles of counting at this moment you