in the previous lecture i have introduced the topic of counting and we discussed little bit of historical aspects of it that is how the subject developed over the past few centuries and then we started discussing the fundamental principles of counting the first principle was the addition principle which simply says that if there are a number of ways of doing one type of thing there are several number of ways of doing another thing then what is the total number of ways doing either of them

so it is simply adding the second fundamental principle of counting is called the multiplication principle in which we are sort of looking at the simultaneous occurrence of two different events

so if one event occurs in m ways and another event occurs in n ways then in how many ways both of them occur together that means either we say first occurs then the second occurs or the second occur then the first occurs or it could be exactly simultaneous also

so then it becomes multiplication that is the total number of phases m into n

so let us continue looking at some of the examples to illustrate this thing

so the next problem is like this how many six digit natural numbers can be formed whose first and third ah digit is ah not first rather fifth and third digit is odd and remaining digits are even ah we assume that zero is also even because g zero can be a digit here

so we assume zero to be ah even here

so let us look at the solution

so the first digit is even

so there are possibilities 2 4 6 8 that is 4 ways are there ok ah the first digit cannot be zero because we are saying six digit numbers

so therefore first digit cannot be zero

so ah since we are saying it has to be even because we have put the condition the third and the fifth digits are odd

so first second fourth and sixth digits they are even

so the first digit even can be chosen in four different ways then the second fourth and sixth digits each can be from either of 0 2 four six eight that means five into five into five ways

so i have applied the multiplication principle here that is the second digit can be chosen in five ways the fourth digit can be chosen in five ways and the sixth digit can be chosen in five ways and all of them have to be together

so therefore five into five into five ways now let us look at the third the third and fifth digit each can be either one three five seven or nine that means five different ways of choosing the third digit and five different ways of choosing the fifth digit that is five into five ways

so now we have fixed the first digit can be chosen in four ways second third fourth fifth sixth each of them can be chosen in five ways

so the total number of ways becomes

so total number of such six digit numbers is four into five to the power five of course you can multiply it out and then see the answer is twelve thousand five hundred such numbers are there ah let me just repeat or review the problem there are six digits we wanted to choose ah this six digit number in particular fashion

so we said that the third and the fifth digit is odd and the remaining digits are even if we do that then in order to choose the first digit we have to choose out of two four six hundred

so there are total number of ways is four now for the remaining each of them can be chosen in five different ways

so therefore the five digits that is the first second ah the second third fourth fifth and sixth digits they can be chosen in five to the power five ways

so therefore the total number of possibilities is four into five to the power five that is there are total such twelve 12500 numbers let us take another problem in a mathematics conference there are n mathematicians ah it turns out that each mathematician discussed with every other mathematician in the conference exactly one problem
 so how many problems were discussed
 so each one discusses with n minus one other mathematicians ah there are total n mathematician each one discusses with the other n minus one
 so it becomes by multiplication principle n into n minus one problems now in this particular counting every person is counted twice because for example if i say mathematician a discusses with mathematician b then i have counted it once now again i am saying mathematician b discusses with mathematician a
 so that will be two times whereas i am putting the restriction that each one discusses with another one exactly one problem
 so that means now each person is counted twice in n into n minus one
 so we will have to divide by two however here everyone is counted twice ah
 so we have actually applied the multiplication principle but with little bit of ah modification in this particular problem ah let me solve another such problem how many 10 digit telephone codes can be created where the first two digits are nine and four and the third digit cannot be zero
 so if you look at this the first place is fixed as nine
 so first digit exactly one way for the second digit you are fixing as four
 so now the third digit is not zero
 so the third digit can be from numbers one two three up to nine
 so nine ways and fourth place up to tenth place you can have either of zero one two up to nine
 so there are ten ways each
 so now these are seven
 so by multiplication principle
 so by multiplication principle the total number of codes is nine into ten to the power seven how many distinct terms are there in expressions for example if i consider say x plus y plus z plus t into say a plus b plus c plus d plus e r say i consider x one plus x two plus x m y one plus y two plus y n z one plus z two plus z t now you can consider see if we look at the product here it is nothing but looking like very similar to a cartesian product of two sets in which the first set has four elements and the second set has five elements because each term will appear with each term in the second case exactly once like you will have x a plus x b plus x c plus x d plus x e similarly then with y a y b etcetera and finally t a t b t e etcetera
 so this is exactly acting as a cartesian product of two sets one is consisting of four elements and the other is consisting of five elements
 so in the first case the total number of terms is exactly the same as in the cartesian product a cross b where is 4 and cartesian product cardinality of b is five
 so cardinality of a cross b is nothing but four into five that is equal to twenty
 so if we extend this argument to the second part of this problem in the second part then you can see if i consider the set e one consisting of m elements the set e 2 consisting of n elements and the set e 3 consisting of t elements then it becomes nothing but the product of e one cross e two cross e three
 so following the reasoning given in one we can say that the number of terms is m into n into t ah now let me introduce ah the next notation that is called factorial notation
 so the notation n factorial represents the product of the first n natural

numbers that is n factorial is equal to 1 into 2 into 3 into n minus 1 into n

so for example we say one factorial is equal to one two factorial is equal to one into two that is two three factorial is equal to one into two into three that is equal to six and

so on as a convention we define zero factorial to be one ah this is to keep consistency with some other notations of permutation and combination that i will be using later on ah in that it will turn out that if we take 0 factorial to be 1 then the consistency of the notation will remain also we do not define the factorial of a negative integer

so it is only for basically positive integers and we include 0 factorial as a special case now something very simple you can observe see this n factorial is equal to n into n minus one factorial ah i had mentioned in the beginning about the historical development of the combinatoric subject ah there are indications that actually this notation was also known to indian mathematicians ah that is the continuous product of natural numbers ah however they did not ah use the notation n factorial etcetera this notation n factorial is actually introduced in 1808 by ah christian cram there are indications that ancient indian mathematicians new ah the concept of product of consecutive natural numbers the modern notation was introduced by christian cram in eighteen hundred eight ah what is important to observe about this ah number ah this notation

so for example one factorial is one two factorial is two three factorial is six you can see that it increases very rapidly

so let me just illustrate it we can note that n factorial increases very rapidly as n increases

so for example if i consider say four factorial it becomes twenty four if i consider five factorial then it becomes five into four factorial that is one hundred twenty six factorial becomes six into five factorial that is becoming seven hundred twenty then seven factorial will become seven into seven hundred twenty

so you can immediately see that number is increasing very fast by seven factorial we have come to five thousand forty and then again you can see if i put eight factorial then again i have to multiply by this ah by eight

so i will get into something like forty thousand plus something

so that is 4 into 10 to the power 3 kind of thing and then again if you say n 9 factorial 10 factorial ah the number is increasing very very rapidly ah however this factorial notation is one of the very very you can say important parts of the current mathematical terminology and in every aspect whether you are doing probability theory whether you are doing ah combinatorics or you are doing calculus there everywhere this factorial notation is widely used now using this we come to the next term that is called the permutation

so what is a permutation an ordered arrangement of k distinct elements from a set of n distinct elements is called a k permutation the total number of all k permutations is denoted by the notation n p r sorry n p k sometimes it is also written like this n p k

so in various books you will see various notations i will be using ah this notation now let us see what would be the value of this that means if we have a set consisting of n distinct elements then how many ordered arrangements of k distinct elements from there can be taken

so please note here i am talking about ordered arrangements suppose i say n is equal to three

so suppose i say a b c suppose i am considering three elements here suppose this is the set a and i want to choose say two here

so i can choose a b i can choose a c i can choose b c but if i am looking at the ordered arrangement then i will also count b a c a and c b

so total such arrangements becomes six such cases

so in general if i am choosing k ordered things from n then how many

so i have given the notation npk let us calculate this thing ah in order to evaluate n p k we can proceed as follows the first element can be chosen in n ways the second element can be chosen in n minus 1 ways and

so on the kth element can be chosen in n minus k plus one vase

so now you apply the general multiplication principle

so by the general multiplication principle the total number of ways is n into n minus one into n minus two and

so on up to n minus k plus one

so that is n p k

so we have developed the formula for npk ok ah we can look at this

so this we can write actually npk as n into n minus one and

so on up to n minus k plus one and i consider multiplication by n minus k n minus k minus one up to three to one and then divide by the same number that is n minus k into n minus k minus one and

so on up to three two one then this number is nothing but if you see the numerator becomes n factorial and the denominator is n minus k factorial

so that is n p k has an alternative expression n factorial divided by n minus k factorial in order that this is well defined you should have one less than or equal to k less than or equal to n ah this makes sense to have actually n also included here that means if you choose all n then how many such things will be there it will become n factorial

so let me just illustrate this again

so if we consider all possible arrangements of n distinct elements then that will become equal to n p n that is n factorial divided by n minus n factorial that is n factorial divided by zero factorial that is n factorial because zero factorial i am taking to be one

so for example if i consider two elements a and b then the number of ways will be a b b a

so that means 2 or 2 factorial if i consider 3 a b c then it will become a b c a c b b c a b a c c a b and c b c therefore total number will become six which is nothing but three factorial because three into two into one ah if we allow the repetition then the answer will change

so the number of permutations of n different objects taken k at a time where repetition is allowed is n to the power k ah because in the first place i can consider n in the second place any n because repetition is allowed

so n into n k times

so it becomes n to the power k now if we consider here i have considered this all n items to be distinct but there is a possibility that some of the items may not be distinct

so in that case this formula can be modified the number of ah distinguishable permutations of n objects of k different types for example where n one objects are ally are similar basically they they should be of same nature actually

so let me not write similar i like n2 are alike and

so on nk are alike and you have n1 plus n 2 plus n k is equal to n then the number of distinguishable permutations is n factorial divided by n one factorial n two factorial and

so on n k factorial ah let me explain the reasoning i consider that if we have all possible arrangements of n distinct elements

so it is n factorial now suppose out of this n objects two are similar are alike and remaining n minus two are distinct now if we are considering any positions where these two items which are alike appear then in whichever order they appear it will not be distinguishable

so that means if i am counting them two times in n factorial then i should divide by two

so the answer will become n factorial by two factorial or two similarly if i say three items are similar then those three items wherever they appear in whichever order that order will not matter because they are similar or alike now we have counted them three factorial times that is one into two into three

so the number will become n factorial divided by three factorial

so now if we extend this argument then if n one things are same then we should divide by n one factorial then another n two type of things are same then we should divide by n two factorial and

so on finally when n k things are same then those n k factorial things also have to be divided because the number of ordered arrangements is same as the unordered arrangement because here that ordering or an ordering does not make any difference because all of the items are alike

so for example just i show through a live thing

so these two pens are there

so you can see here they are alike

so whether i put this here and this here or i put this here and this here it does not make a difference but if i consider this blue and this black then if you see if i put this first and this here that is the left and right and if i change the order then these are two different arrangements

so now if you consider three now if i have three distinct things

so let me just take these three distinct things then i can have one two three

so that is i can put this here this is another arrangement i can put this here this is another arrangement and this i can put here this is another arrangement then i can put this here this is another arrangement and this i put here this is another arrangement

so total six arrangements are there but if two of them are alike

so if i take this now let us look at how many arrangements will be there this is here or this is here

so there are total three arrangements now three distinct arrangements are there why because this two in whichever order you keep it does not make any difference

so that means three factorial has been divided by two that is six divided by two that gives you the answer three

so therefore you have this ah general formula that means if i have n distinct items out of which n1 are alike n2 are alike and nk are alike then the total number of distinguishable permutations is n factorial divided by n one factorial n two factorial and n k factorial that is coming in the denominator

so ah i hope i have made the explanation very clear let us look at some examples here how many different 10 letter codes can be generated using two a's three b's two c's and three d's ah there are total ten letters available with me that is two a's three b's two c's and three d

so total ten numbers are there as you can see here there are two of a

so in whichever order i put it will not make any difference there are three b's in whichever order i will put that will not make any difference similarly about two cs and similarly about three ds

so the total number of arrangements by this formula will be distinct codes that is 10 letter all 10 are used here that is equal to 10 factorial divided by 2 factorial 3 factorial two factorial three factorial

so after some simplification this number can be evaluated it is twenty five thousand two hundred codes that are possible ah let me take another problem here how many words can be made using all letters of the word nation ok

so let us look at this if i have to use all the six letters then you look at how many are distinct here there are a i o n and t that is occurring

so 5 distinct letters is there out of which n is repeated once
so here we have six letters where n occurs twice
so the total number of arrangements of these six letters by the formula it will become six factorial divided by two factorial
so that is equal to three hundred sixty ah similar problem let me look at how many distinct 11 letter arrangements of the word probability can be made
so you can see here in probability we have eleven letters
so total number of letters is eleven out of this if you observe b occurs twice and also i this i also occurs twice
so the total number of arrangements by this formula it will become eleven factorial divided by two factorial into two factorial ah this number can be evaluated it is ah ninety nine lakhs seventy nine thousand two hundred ah as you can see here this eleven factorial is a huge number
so that is four times of this which we divided by four
so we are getting
so it is actually already coming in ah tens of millions actually
so ah that is the nature of this number factorial it increases ah very very rapidly now in this itself if we observe in the probability there are some vowels and some consonants are there
so suppose we want to look at that arrangements also
so let me just pose a problem here in how many of these arrangements say vowels appear together
so let us look at this here we are having seven consonants in which b is repeated twice ok
so now if we consider all the vowels to be together that means whenever they occur they appear together then actually there are basically eight items here that is the seven consonants and all the vowels are together
so we can treat this as one item since all four vowels where i is repeated have to appear together we have total eight items
so the number of arrangements is 8 factorial divided by 2 factorial because b has been repeated ah now there is a catch here these vowels although they are taken together but they can permute among themselves
so therefore how many such arrangements are possible ah since vowels can be permuted among themselves such arrangements are 4 factorial divided by 2 factorial because i has been repeated here
so that is nothing but ah twelve
so now the final number will be equal to
so total number of words from the letters of word probability where vowels occur together that will become eight factorial divided by two factorial into twelve
so of course you can evaluate it it is equal to twenty four ah two lakh forty one thousand nine hundred twenty ah what we have done we have calculated all the words which are created from the word probability we have also calculated the arrangements where the vowels appear together
so there is an additional problem you can solve from here suppose we say vowels do not appear together
so in how many of these vowels do not appear together then it will be from total number of arrangements minus the number of arrangements in the which the vowels are together
so you take the answer of question number three minus the answer of question number four
so that number will become ninety nine lakh seventy nine thousand two hundred minus two lakh forty one thousand nine hundred twenty that is ninety seven lakhs thirty seven thousand two hundred eighty

so let me just solve another problem of similar nature in how many ways letters of the word say towel can be arranged

so that all vowels occur together and all consonants occur together

so let us observe this ah how many consonants are here there are three consonants all are distinct that is t w and l similarly there are two vowels o and e

so they are also distinct

so basically if i am saying all the consonants appear together then it is treated as one entity and two vowels occur together then that is also considered as one entity

so basically arrangement of them can be either firstly all the consonants then all the vowels are reverse of it

so basically two possibilities are there but now you look at this consonants themselves can be permuted and they are all distinct

so there are three of them

so it becomes three factorial similarly all the vowels can also be permuted among themselves

so that is two factorial

so you will have ah the number of ways will be 2 into 3 factorial into 2 factorial

so that is equal to 24

so there are total 24 letters of t o w e l where i all the consonants are appear together and all the vowels appear together

so let me just repeat here in the topic of permutations we are actually looking at the ordered arrangements of the items that means if i am considering ah positions then they have to be fixed if i interchange the positions of the items then it is considered another arrangement now if we exclude that thing that means if i consider this arrangement and this arrangement as the same that means i consider unordered arrangement then it is called combination

so i give a new definition now an unordered arrangement of k distinct items from a set of n distinct items is called a k combination the total number of all k combinations is denoted by n c k or there is another notation popularly used n c k sometimes it is written as n c k like this also

so in various books you will have different notations i will generally use n c k like this or like this again you can see here one is less than or equal to k less than or equal to n ah now let us find out the value of this in order to evaluate say x that is equal to n c k we proceed as follows

so if we consider the ordered arrangements there are n p k the number of ordered k arrangements is n p k ah now for this k things there are k factorial such things but now if we say that ordering is not important then all of this things will be considered same

so for example if i say a b

so a b and b a they are distinct if i am considering ordered permutation but if i consider unordered then a b and b a are the same

so that means you have to consider number as two divided by two similarly if i consider three things

so a b c b c a and

so on

so i have three factorial such things now i will be considering all of them as one

so i have to divide the number as divided by three factorial

so if i look at this there are k factorial permutations now all of this will be considered same

so of these k objects

so you should have n p k divided by k factorial is equal to x that is n c k
so this is the formula that now you got n c k is nothing but n p k divided by k factorial

so this implies that n c k is nothing but n factorial divided by k factorial n minus k factorial in fact i can consider 0 also because 0 factorial we have defined a historical notice that historically um this type of formula appears in the work of indian mathematician bhaskarachar ii is timeline is around 114 to 1185 a d ah let us look at some of the properties of this combination actually this terms npk and nck they have lot of relations and there are several properties

so i will cover some of these here

so for example n c k is equal to n c n minus k the proof is extremely simple because it is just based on what you are writing

so nc n minus k is equal to n factorial divided by n minus k factorial n minus n minus k factorial that is equal to n factorial divided by n minus k factorial into k factorial which is nothing but again n c k ah

so friends actually we have discussed four major counting principles one is the ah fundamental principle of addition second is the multiplication principle third one is the total number of ordered arrangements that is we are denoting by permutation n p k and the fourth one is the number of unordered arrangements that we are denoting by nck ah in the next classes i will continue to describe the properties of this and various problems that are based on this i will be solving in the next few lectures you