welcome students welcome to the series of lectures on matrices and determinants in the last few lectures we were trying to solve a system of linear equations by reducing it to the to its row reduced echelon form in this lecture we will see some more problems on solving a system of linear equations and problems based on it

so lets start with the problem solve the system a plus b equal to eight a plus c equal to thirteen b plus d equal to eight and c minus d equal to five solution as usual lets first write down the matrix form 1 1 0 0 1 0 1 0 0 1 0 1 0 0 1 minus 1 and this acts on the unknown vector or the unknowns a b c d will get the constant of matrix which is 8 13 8 and 5.

let us write down the augmented matrix of this 1 1 0 0 augmented with 8 1 0 1 0 0 1 0 1 0 0 1 minus 1 augmented with 8 13 8 and 5.

now let us try to convert this into the into its rre

so only the coefficient matrix part

so the first row the first element is one

so let us not bother about it

so we will have to convert this other element which is one which is in the second row into zero

so r two is replaced by r two minus r one

so the resulting matrix is the first row remains unchanged one one zero zero augmented with eight second row it is r two minus r one one minus one which is zero zero minus one you will get minus one one minus zero which is one zero minus zero which is zero thirteen minus eight which is five other rows remains unchanged zero one zero one zero zero one minus one eight and five they remain unchanged as it is now the other one next row the second row we have minus one that is the first non-zero element will have to convert that minus one into one

so let us multiply the second row by minus one

so r two is replaced by minus half r one one zero zero augmented with eight the second row we have just multiplied by minus one

so zero one minus one zero minus five the other two rows remains unchanged zero zero one minus one five now we have one and one on the first and second row sorry first and third row on the second column

so let us convert them into zero

so r one is replaced by r one minus r two similarly r three is replaced by r three minus r r one minus r two

so one minus zero which is one one minus one you have zero zero minus minus one you have one zero minus zero you have zero eight minus minus five you have eight plus five which is thirteen second row remains as it is third row is replaced by r three minus r two zero minus zero you have zero one minus one you have zero zero minus minus one you have one one minus zero one eight minus minus five which is eight plus five you have thirteen last row remains as it is now that you have one over here

so let us convert the other elements which is which are one minus one and one into zeros r one is replaced by r one minus r two r two is replaced by r two sorry r one minus r three first one second one r two is replaced by r two minus r two plus r three third one r four is replaced by r four minus r three first one first row is r one minus r three one minus zero you have one zero minus zero you have zero one minus one you again have zero zero minus one you have minus one and finally thirteen minus thirteen you have zero second one r two plus r three zero plus zero you have zero one plus zero you have one minus one plus one it is zero zero plus one you have one minus five plus thirteen you have eight third row remains as it is zero zero one one 13 last row it is r 4 minus r 1 0 minus 0 0 0 minus 0 0 1 minus 1 it is 0 minus one minus one you have minus two five minus thirteen

so you have minus eight you have minus two

so let us convert it into one r four is replaced by minus one by two times r four one zero zero minus one and zero zero one zero one eight zero zero one one thirteen zero zero zero one and you have a two sorry you have a four because minus eight by minus two you have four

so let us convert the other elements in the last column which are one one and minus one into zeros r one is replaced by r one plus r four r two is replaced by r two minus r four r three is replaced by r three minus r four now let us do these operations one minus zero one zero minus zero zero zero minus zero zero sorry one plus zero it is one zero plus zero zero plus zero minus one plus one it is again zero zero plus four you have four second one r three r two minus r four

so zero minus zero zero one minus 0 1 0 minus 0 it is 0 1 minus 1 it is again 0 8 minus 4 you have 4 0 minus 0 you have 0 0 minus 0 1 minus 0 1 one minus one you have zero thirteen minus four you have nine the last row remains as it is zero zero zero one and you have four thus the solutions that we have are the following a equal to four b equal to four c equal to nine and d is four this is the solution now let us do the second problem solve the system x minus three y plus two z equal to zero two x minus five y minus two z equal to zero four x minus eleven y plus two z equal to zero let us write down the solution to this let us first write down the augmented matrix one minus three two two minus five minus two four minus eleven and two now let us try to convert this into its r r e r two is replaced by r two minus two times r one r three is replaced by r three minus four times r one let's do the following operations the first row remains unchanged the first column will have zeros now let us do the things minus five minus two times minus three

so you will have minus five plus six

so you will have one minus two minus four you will have minus six minus eleven plus twelve

so you will have one two minus eight

so you will have minus six now let us convert minus three and one into zero r one is replaced by r one plus three times r two and r three is replaced by r three minus r the first column remains unchanged similarly the second column you can notice that it is going to be 0 1 0 r 1 2 plus 3 times minus 6 which means 2 minus 18 you will have minus sixteen minus six and last one r three minus r two minus six minus minus six you will have zero now notice that you have a zero column and therefore rank of the coefficient matrix only two

so we have independent and dependent variables

so now you do not have a one or the leading coefficient in the last variable

so we will treat the last variable that is that as the that is the independent variable

so let z equal to lambda

so the first equation that we have let us write down x minus 16 times z is zero which means x is this will imply that x equal to sixteen times lambda the second one will give me y minus six z as zero that will imply that y equal to six times lambda therefore the solution is the following 16 lambda 6 lambda lambda with lambda n r right these are all the solutions which means that this system has got infinite number of solutions now let us do the next problem solve the system t minus u plus two v minus three w equal to nine four t plus eleven v minus ten w equal to forty six three t minus u plus eight v e minus six w equal to twenty seven as usual let us write let us start with writing the augmented matrix solution the augmented matrix is one minus one two minus three augmented with nine 4 you have 0 11 minus 10 46 3 minus 1 8 minus six and twenty seven now let us try to convert this into its rre you have one

so lets not bother you have four and three lets convert them into zeros r two is replaced by r two minus four times r one r three is replaced by r three minus three times r one let us do the following operations the first row remains unchanged second row you have 0 r2 minus 0 minus four times minus one which is zero plus four you have four eleven minus two times four which is eleven minus eight you have three minus ten minus four times minus three which means minus ten plus twelve you have two forty six minus four times nine which means forty six minus thirty six you have ten next one r three minus three times r one which will give me zero here next one minus one minus three times minus one which is minus one plus three you have two eight minus 3 times 2 which is 8 minus 6 which will give me 2 minus 6 minus 3 times minus 3 which means minus 6 plus 9 which will give me 3 seven minus three times nine which will give me just zero i have four let us divide this by four r two is replaced by one by four times r two first row remains unchanged second row i am dividing it by four zero one three by four half ten by four which is five by two last row remains unchanged let us convert the other elements which are minus one and two into zeros r one is replaced by r one plus r two similarly r three is replaced by r three minus two times r two you have one zero zero zero one zero now let us do the remaining things two plus three by four which means eleven by four minus three plus half which will give me minus five by two nine plus five by two which will give me twenty three by two second row remains as it is three by four half phi by two last row r three minus two times r two

 so two minus three by two two times

 so two minus two times three by four which will give me two minus three by two which will just lead to half three minus two times half which is three minus one which will give me two zero minus two times five by two which will give me minus five

 so now that we have just half here let us convert it in let us multiply this row by two and make it into one r three is replaced by two times r three the first two rows remains unchanged zero zero one four minus ten lets convert the other element three by four and eleven by four into zeros r one is replaced by r one minus eleven by four times r three r two is replaced by r two minus three by four times r three the first three columns are just going to be zero one zero zero zero one zero and zero zero one now let us try to manipulate the remaining columns minus five by two minus eleven because you have eleven by four into four which is just eleven twenty three by two minus eleven by four into minus ten

 so you will have plus fifty five by two second row half minus 3 by 4 into 4 which will just give me 3 5 by 2 minus fifteen by two the last row remains as it is now let us compute these the resulting matrix is one zero zero zero one zero zero zero one minus y by two minus eleven

 so you have minus twenty seven by two next to 1 minus 5 by 2 4 this is 78 by 2 which is just 39 minus ten by two

 so you have minus five last one is just minus ten in the last column there is no one

 so let us call it as the independent variable that is w is the independent variable

 so let w equal to lambda and therefore the first equation will give me t minus twenty seven by two times w is just thirty nine that will imply that t is twenty seven by two times lambda plus thirty nine second one u minus five by two times w is just minus five that will imply that u is five by two times lambda minus five v plus four times w is minus ten that will imply that v is minus four times lambda minus therefore the general solution is twenty seven by two times lambda plus 39 phi by 2 times lambda minus 5 minus 4 lambda minus 10 and lambda that's lambda any being any real number let us do one more problem on the similar lines

solve the system x plus two i plus three z equal to one two x plus y plus three z equal to two five x plus five y plus nine z equal to four solution let us begin by writing the augmented matrix 1 2 5 2 1 5 3 3 nine augmented with one two and four now let us try to convert these two elements two and five into zero because the first element is just one r two is replaced by r two minus two times r one and r three is replaced by r three minus five times r one the first row is going to be just one zeros first column is going to be just one zero zero right second column r two which is one minus two times two which is one minus four first column first row remains as it is two one two three and one

 so one minus four which will give me minus three three minus two times three which is three minus 6 which will give me minus 3 2 minus 2 into 1 which is 2 minus 2 which will give me 0 now r 3 minus 5 times r 1 5 minus five into two which is five minus ten which will give me minus five nine n minus five into three which is nine minus fifteen that will give me minus six four minus five into one which will give me four minus five which is just minus one

 so i have a minus three lets divided by minus three r two is replaced by minus one by three times r two first row remains as it is one two three and one zero one one zero last row again remains the same zero minus five minus six and minus one now let us convert this two and minus five into zeros r two is replaced by sorry r one is replaced by r one minus two times r two and r three is replaced by r three plus five times r

 so the first and second column are going to look like one zero zero and zero one zero now let us do the same set of operations on the third and fourth column r one three minus two times one which is three minus two which will give me one one minus two times zero which is one minus zero which will have one second row remains as it is r three plus five times r two minus x plus five into one which is minus six plus y five which will give me just minus one the last one is just minus one now let us convert this minus one into one r three is replaced by minus r three therefore i will have one zero one one zero one one zero zero zero one and one now let us convert this one and one into zeros r one is replaced by r one minus r three and r three is replaced by r three minus sorry r two minus r three we have the identity matrix one zero zero zero one zero and zero zero one finally one minus one that is just zero zero minus one you have minus one and one right thus the solutions are x equal to zero y equal to minus one and z equal to one this is the solution now let us do one more problem

 so let us try to solve the following system solve the system x plus i y equal to zero minus i x plus z equal to zero and y minus z equal to zero you can notice that this is a system with complex coefficients this is a system with complex coefficients let us write down the augmented matrix one i zero minus i zero one zero one minus one right this is the matrix now let us try to convert this into its r r e r two is replaced by r two plus i times r one the first row remains as it is one i zero here you have zeros and zeros now second one zero plus i times i which is i square you have minus one next one remains as it is last row remains as it is now let us try to convert this minus one into one just by multiplying the second row by minus one r two is replaced by minus of r two will have one i zero zero one minus one zero one minus one let us convert this i and z one into zeros r one is replaced by r one minus i times r two and r three is replaced by r three minus r two

 so the first and second column are going to look like one zero zero and zero one zero now let us do the remaining calculations r one zero minus i times minus one which will just give me i second row remains as it is third one is going to give me zero

 so you have a zero row which means that the rank of this coefficient matrix is just two and therefore you have an independent variable and

so the independent variable is just the last one which is z variable because there is no pivot element

so that is the independent variable

so what i have now let us write down z as lambda now let us write from the equations write down from the equations x plus i times z 0 that will give me x is minus i lambda second one y minus z 0 that will give me y equal to lambda therefore the solution set is given by minus i lambda lambda lambda with lambda n r which means you have an infinite number of solutions let us look at one more problem determine the values of lambda and mu for which the system x plus two i plus three z equal to six x plus three y plus phi z equal to nine two x plus five y plus lambda z equal to nu has number one no solution number two unique solution and number three infinite number of solutions right

so you are given the system with two unknowns lambda and mu

so you have to find the values such that this system has got no solution unique solution and infinite number of solutions let us try to write the augmented matrix one two three six one three five nine two five lambda mu this is the matrix lets do the following operations r one is replaced by r three minus two times r one and r two is replaced by r two minus two times r three minus two times r two

so that what you will have

so r three minus two times r one

so two minus two into one

so you will have zero zero two let us compute the other one five minus two into two which is five minus four you will have one five minus three into two which is five minus six you have minus 1 5 lambda minus 3 into 2 which is lambda minus 6 lambda minus five into two which will give me lambda minus five into two which is ten last one is just lambda last one mu minus two into six which is mu minus twelve mu minus nine into two which is mu minus ninety sorry eighteen and then you have mu now let us replace r one by r one plus r two first column is zero zero two and then you have zero minus one five this plus this

so what you will have is lambda minus sixteen lambda minus ten lambda if you had sorry you have two lambda minus sixteen you have two mu minus thirty mu minus eighteen now let us write down the first one 2 lambda minus 16 times z is two mu minus thirty or equivalently lambda minus eight times z equal to mu minus fifteen now let us consider the following cases case one if lambda equal to eight and mu n any real number minus just fifteen then what will happen you can notice that if lambda is eight and mu in r minus fifteen which means you have zero over zero in the first row pertaining to the coefficient matrix but there is going to be a non-zero term on the right hand side which means the system has no solution system has no solution second one if lambda is not equal to eight which means this term is a nonzero quantity and mu is any real number once you have a non zero quantity right whenever lambda is not equal to eight you can notice that all the three terms are non zero which means that the coefficient matrix has full rank and for any mu you can always notice that this system also has got rank three and therefore the system has a unique solution and finally if lambda equal to eight and mu equal to fifteen if both these cases arises which means the first row completely becomes zero in this case the system has infinite number of solutions right you have this much first case is lambda equal to eight and mu any real number apart from fifteen in this case the system has no solution second case lambda not equal to eight and mu any real number in this case the system has a unique solution and the last case when lambda equal to 8 and mu is 15 in this case the system has infinite number of solution now let us proceed towards next problem if the system x plus a y equal to zero a z plus y equal to zero and a x plus z equal to zero has infinite number of solutions then

find the value of a solution let us try to write down the coefficient matrix because you have only the 0s on the right hand side 1 a 0 0 1 a a zero one this is the coefficient matrix let us try to convert them into its rre before converting notice that if a equal to zero then the system has a unique solution which is just zero right because if a is zero what will end up is just the end up with is just the identity matrix therefore in this case the solution is just 0 0 0 but what we wanted us the value of a for which the system has got infinite number of solutions and therefore a is equal to zero is not allowed
 so let us assume that a not equal to zero now let us try to convert it into its r r e r three is replaced by r three minus a times r one the first row column is one zero zero second first row in fact remains unchanged even the second row a zero one a third one zero minus a times a you will have minus a square one minus a times zero you have one you have one
 so let us convert the other two elements a and minus a square into zeros r one is replaced by r one minus a times r two and r three is replaced by r three plus a square times r two first row one a minus a you have zero zero minus a square
 so you have minus a square second row remains as it is third row zero minus zero zero one plus a square into a which is one plus a cube now for these system to have infinite number of solutions what is expected is at least one zero row now you have two zeros for the last term to become zero you need one plus a cube to be zero which means a cube should be minus one
 so what we expect is one plus a cube to be zero because for the other two rows you already have an one
 so you have a non zero term
 so which means you cannot expect them to be ah to become a zero zero row
 so only thing that you can expect is the last one you already have two zero
 so last term which is one plus a cube that can become zero if that becomes zero that will imply that a cube is minus one that is a equal to minus one
 so if a equal to minus one the system the coefficient matrix has got rank two therefore the system has got has infinite number of solutions if a equal to minus one before we proceed to the next one let us define what is known as definition a square matrix a is said to be orthogonal if a a transpose equal to identity right if whenever you multiply a with its transpose then what you should end up with an identity matrix if such a thing happens then you say that such a matrix is an orthogonal matrix now the next problem is if a equal to 0 alpha alpha 2 beta beta minus beta gamma minus gamma gamma is an orthogonal matrix then find the values of alpha beta and gamma let us try to solve this what we have is a a transpose equal to identity let us write down 0 alpha alpha 2 beta beta minus beta gamma minus gamma gamma when multiplied with 0 alpha alpha now i am writing it row wise 2 beta beta minus beta gamma minus gamma gamma when you multiply these two what you should end up with is the identity matrix three by three identity matrix one zero zero zero one zero and zero zero one fine now let us write let us multiply the matrices that are in the on the left hand side four beta square plus gamma square two beta square minus gamma square minus two beta square plus gamma square sorry second column two beta gamma sorry two beta square minus gamma square alpha square plus beta square plus gamma square alpha square minus beta square minus gamma square last one minus two beta square plus gamma square alpha square minus beta square minus gamma square alpha square plus beta square plus gamma square this should be same as one zero zero zero one zero zero zero one now let us write down the equations four beta square plus gamma square equal to one second one two beta square minus gamma square equal to zero third one
 so this is same as the second fourth one is again same as this next one alpha square plus beta square plus gamma square should be one alpha square minus beta

square minus gamma square should be zero these are the four equations that i have now let us try to solve them let us first write down in the form of a augmented matrix

 so here here we have four equations with alpha square beta square gamma square as the variables

 so zero four one augmented with one zero two minus one zero one one one one one minus one minus one and zero this is what we have let us try to convert them into its r e

 so what we will rather do is we will swap r one and r three and similarly will swap r two and r four you have one one one augmented with one and then one minus one minus one zero and then you have zero four one one zero two minus one zero now let us convert them into let us convert this one into zero r two is replaced by r two minus r one first row remains as it is second row r two minus r one you have zero minus one minus minus one minus minus one which is minus two again minus two zero minus one which is one third and fourth row remains as it is r two minus zero minus one you have minus one over here now let us convert this minus two into one r two is replaced by one by minus two times r two the first column remains unchanged first row also remains unchanged second row you have 0 1 1 and then half third and fourth remains unchanged now let us convert this one four and two into zeros r one is replaced by r one minus r two similarly similarly we have r three is replaced by r three minus four times r two r four is replaced by r four minus two times r two lets compute all these things r one minus r two first column remains one zero zero second column sorry one zero zero zero second column zero one zero zero now let us do the other one r one minus r two you have zero 1 r 3 1 minus 4 times r 2 which means 1 minus 4 you will have minus 3 minus 1 minus 2 times r two which will just give me minus three again r one minus r two one minus half which will give me half second row is just half third row r three which is one minus four times half which is one minus two which will give me minus one minus two sorry one minus two one minus two ah minus one zero minus two times half which will give me minus one lets convert this minus three into one r is replaced by one by minus three into r three

 so you have one zero zero zero zero one zero zero zero one one minus three and then you have half half one by three minus one let us convert this one and minus three into zeros r two is replaced by r two minus r three r four is replaced by r four minus sorry r four plus three times r three one zero zero zero one zero zero zero one and then you will have zero zero zero let us compute the last column r two minus r three sorry first one is just half r two minus r three

 so one by two minus one by three which will end up with one by six three minus two which ah one by six and then you have one by three last one is going to be zero now let us write down the solution

 so the solution in this case is alpha square is one by two beta square is one by six and gamma square is one by three thus the values of alpha beta and gamma are alpha equal to plus or minus one by root two beta equal to plus or minus one by root six and gamma equal to plus or minus one by root three right

 so these are all the values of alpha beta and gamma for which the given matrix becomes the becomes an orthogonal matrix

 so with this let me stop thank you all you