welcome students to iitpal problem solving session of mathematics our topic is probability and this is lecture number six as i said at the end of the last class today we shall look at binomial distribution and use of binomial theorem in counting if you remember in the very first lecture i was talking about tossing a coin three times and noting the sequence of outcomes and we have seen that the sequences t t t it can be t t h t h t h t t h h t h t h t h h and all three are h thus there are eight possible sequences when we are tossing a coin three times the problem of such sequences is that these are not manipulable using mathematics and more importantly suppose there are 100 tosses and therefore there will be 2 to the power 100 many sequences that is too large to handle therefore what we bring in is called the concept of random variable for you i give a very simplified definition of random variable which is as follows a random variable is a mapping from the sample space omega to real number this is not the complete definition but at your level let us stick to this now suppose with respect to the earlier experiment consider x to be a random variable such that x of omega is equal to number of heads in the sequence therefore since we have these eight sequences t t t t t h t h t h t t h t h h h t t h h and h h h

so these are my omegas therefore if we apply x then what we get there are 0 heads in this these 3 leads to 1 these 3 leads to 2 and this leads to 3 therefore the random variable x takes values either 0 or 1 or 2 or 3.

notice that x takes values 0 1 2 3 when number of tosses is equal to therefore if number of tosses is n then corresponding random variable x will take values 0 1 up to n because there can be no head one heads up to all n tosses giving head another thing notice that the probabilities of i i belonging to 0 1 up to n is not same for all i as you can understand that with respect to x when it is taking values 0 1 2 3 then number of cases for 0 is 1 number of cases for 1 is equal to 3 number of cases for x is equal to 2 is 3 and number of cases for 3 is equal to 1.

so suppose p is the probability that the coin a head in one toss therefore probability of t t t is equal to 1 minus p into 1 minus p into 1 minus p is equal to 1 minus p whole cube which is often we write q cube similarly probability of one head is equal to probability of t t h plus probability of t h t plus probability of h t t is equal to q q p plus q p q plus p q q is equal to three times p q square similarly of two h is coming out to be three p square q and probability of three h is equal to p cube we can see that the probabilities can be obtained by q plus p whole to the power 3 and probability that x is equal to y is therefore going to be out of 3 you choose i positions put a head there which can be done in this way and then in the remaining positions you put q in general if a coin has probability p for head then the random variable x giving the count of heads in n tosses can be obtained from q plus p whole to the power n and probability x is equal to i is equal to n c i p to the power i q to the power n minus i this is called the distribution or probability mass function for binomial distribution with parameters n and p which we often write as binomial in p

so let us solve a few problems suppose we have x is a random variable following binomial 6 comma p it is given that nine times probability of four is equal to probability of two what is the value of p answer probability of 4 is out of 6 tosses we are getting 4 heads that is 6 c 4 p to the power 4 q to the power 6 minus 4 that is 2 and probability of 2 is equal to 6 c 2 p to the power 2 q to the power 4 therefore p 2 that is probability of 2 upon probability of 4 is equal to 9 it is given that is 6 c 2 p square q to the power 4 upon 6 c 4 p to the power 4 q square is equal to 9 that is q square upon p square is equal to nine that is q by p is equal to three because it cannot be negative all

probabilities are positive therefore q is equal to 3 p now p plus q is equal to 1 therefore p plus 3 p is equal to 1 implies p is equal to 0.
25 or 1 by 4
so that is the answer another problem suppose x follows binomial n comma p when n is equal to eight probability of one is equal to zero point two zero four eight and probability of two is equal to 0.
1024 find the value of p therefore probability of one is equal to eight c one p to the power one q to the power 8 minus 1 that is 7 is equal to 0.
2048 probability of 2 is equal to h c 2 p square q to the power 6 is equal to 0.
1024 therefore by dividing we have 8 p q to the power 7 upon 8 c 2 which is is equal to factorial eight upon factorial two to factorial six p square q to the power six is equal to zero point two divided by 0.
1024 or 8 p q to the power 7 divided by factorial eight upon factorial six
so that is seven into eight that is fifty six divided by two is equal to twenty eight p square q to the power 6 is equal to 2 or 2 upon 7 q by p is equal to 2 that is q is equal to 7 p therefore value of p is equal to 1 upon 8 that is the answer now an important concept of a random variable is its mean or often it is called expectation you know that the mean of a binomial n comma p random variable is equal to n p and its variances n p q this you must have done in your textbooks or your classes
so i am not deriving this formula let us do some problem if the mean of a binomial random variable is four and its variances half of its mean then compute the probability that the random variable takes value greater than equal to two
so answer mean is equal to np variance is equal to npq and given that npq is equal to half np implies q is equal to half therefore p is equal to half therefore mean is equal to 4 implies n into half is equal to four implies n is equal to eight therefore we can get the complete distribution that it is binomial eight comma half we need to compute probability this random variable binomial 8 comma half random variable is greater than equal to 2 that is 1 minus probability of 0 minus probability of 1 is equal to 1 minus 8 c 0 half to the power 0 1 minus half to the power eight minus eight c one half to the power one into half to the power eight minus one is equal to seven is equal to one minus raised to zero is equal to one half to the power eight minus eight c one is equal to eight into half to the power eight is equal to 1 minus 9 times half to the power 8 is equal to 1 minus 9 times 1 upon 2 56 is equal to 256 minus 9 upon 256 is equal to 247 upon 256
so that is the probability that the random variable will take value greater than equal to two
so these are some simple problems on binomial distribution and with that let me go to discussing use of binomial theorem in counting the number of certain events for illustration consider the problem that in how many ways you can have five different positive integers n one n two n three n four and in five such that sigma k ni i is equal to one to five is equal to twenty in fact we have found that the number of solutions is 7 what are those they are 1 2 3 4 and 10 1 2 3 5 and 9 1 2 3 6 8 1 2 4 5 8 1 2 4 6 7 1 3 4 5 7 and 2 3 4 5 6.

so these are the seven possible solutions but we have done it in a mechanical way as you can understand that first i have given maximum weight to the last one and then i try to distribute that weight keeping in mind that all five have to be distinct and that way we have got the solutions the problem is such a solution is not guaranteed to give all possible combinations therefore i want to give you a mathematical way of obtaining the number of solutions i am not going to give you the total exact solutions but the total number of solutions which in this case is 7 how to do that the underlying concept is generating function

which is essentially a power series which may be finite or in finite
so let me first give you some example in how many ways we can have two variables x and y such that 0 less than equal to x less than equal to 2 and one less than equal to y less than equal to two and x plus y is equal to three this solution is pretty simple
so let us first draw a table x y and let us put their sum x can takes value 0 1 2 and y can take value only 1 and 2 therefore we have these 6 combinations and the sums are 1 2 2 3 3 4.
therefore we can see that there are two cases when the sum is three two cases in the sum is two one case of one and one case of four
so how to do that consider the following two polynomials and compute their product
so what are those polynomials because x takes the value 0 1 2 we write z to the power 0 plus z to the power 1 plus z to the power 2.
and the second one is corresponding to y which takes values only 1 and 2 therefore we write z to the power 1 plus z to the power 2 is equal to 1 plus z plus z square multiplied by z plus z square and if we compute the product we get z plus z square plus z cube plus z square plus z cube plus z to the power 4 is equal to z plus 2 z square plus 2 z cube plus z to the power 4.
now can you find a similarity between this product and these numbers and cases we have only one with z to the power 1 2 with z square 2 with z cube and 1 with z to the power 4 thus this polynomial gives the number of solutions for any combination adding up to say 1 2 3 and 4.

so this is the trick that allows us to solve many problems let me give you another example suppose we want to find out x plus y plus z is equal to ten when zero less than equal to x less than equal to 4 y greater than 0 that means it can take any positive integer and z greater than equal to 0 that means it can take 0 also therefore we want to find out the number of solutions such that x plus y plus z is equal to ten
so by going in a similar way therefore we can write the following three polynomials because x takes the value between 0 to 4
so let us write them as x to the power 0 plus x to the power 1 plus x to the power 2 plus x to the power 3 plus x to the power 4 multiplied by y can take any positive values
so that gives us an infinite series x plus x square plus x cube up to infinity multiplied by corresponding to z because z starts from 0 we have 1 plus x plus x square up to infinity the theory is that if we compute the product and obtain the coefficient of x to the power 10
so that will give the number of possible solutions to the problem
so let us try we have 1 plus x plus x square plus x cube plus x to the power 4 multiplied by x if we take out x then we get 1 plus x plus x square up to infinity whole square
so let us try to find out the coefficient of x to the power ten in this
so we have let us write the first one as x plus x square plus x cube plus x to the power four plus x to the power five multiplied by one plus x plus x square whole to the power two therefore we see that if we get the term for x to the power 5 that will be multiplied with this x to the power 6 that will be multiplied by this and x to the power nine which will be multiplied by this and that is going to give us the total number of ways of getting x to the power ten now we know that 1 plus x plus x square whole to the power n is equal to 1 upon 1 minus x whole to the power n that is is equal to 1 minus x whole to the power minus n which can be written as 1 plus sigma over r is equal to 1 to infinity n minus 1 plus r choose r x to the power r

so we are going to utilize this formula and try to solve this therefore we need to find out coefficient of x to the power nine x to the power eight up to x to the power five

so let us first consider x to the power five here n is equal to 2 r is equal to 5 therefore coefficient of x to the power 5 is equal to 2 plus 5 minus 1 c 5 that is the coefficient of x to the power 5 because our formula is in plus r minus 1 c r is equal to 7 minus 1 that is 6 c 5 is equal to 6 let us just try for x to the power nine therefore here r is equal to nine n is equal to 2 therefore coefficient of x to the power 9 is equal to 9 plus 2 minus 1 c 9 is equal to 10 c 9 is equal to 10 i am not solving this but therefore you can understand that the total number of possible solutions is equal to coefficient of 9 which is 10 coefficient of x to the power 9 which is 10 plus in a similar way it is going to be plus 9 plus 8 plus 7 plus coefficient of x to the power 5 which you have computed is equal to 6 this is 19 plus 28 27 plus 7 34 plus 6 that is 40.

therefore the possible solutions is forty note that we are forming the power series corresponding to each variable by considering the values that it may take for example suppose we have the problem in how many ways we can get x plus y plus z is equal to 50 such that x is a multiple of two y is a multiple of three and z is positive and multiple of five these are we are not given whether they are positive therefore x and y can take the value zero also therefore the number of solutions for x plus y plus z is equal to 50 with the above restriction will be given by coefficient of x to the power 50 in 1 plus x square plus x to the power 4 multiplied by 1 plus x cube plus x to the power 6 plus x to the power 9 multiplied by x to the power 5 plus x to the power 10 plus x to the power 15 and if we multiply this 3 power series and obtain the coefficient of x to the power 50 that is going to be the solution to this problem now let me solve problems that we have done in the class again in a mechanical way

so let us recollect this problem our problem is that 0 less than n 1 less than n 2 less than n 3 less than n 4 less than n 5 and it is given that sigma n i i is equal to one two five is equal to twenty in how many ways you can get it therefore we start as follows in order to maintain that these are all distinct we do the following let m two is equal to n two minus n 1 m 3 is equal to n 3 minus n 2 m 4 is equal to n 4 minus n 3 and then 5 is equal to n 5 minus n 4 therefore n one and m two m three m four m five are all greater than 0 moreover n 1 plus n 2 plus n 3 plus n 4 plus n 5 is equal to 20 this we can write as 5 n 1 plus 4 into minus n 1 plus 3 n 3 minus n 2 plus 2 n 4 minus n 3 plus n 5 minus n 4 is equal to 20 or 5 n 1 plus 4 m 2 plus 3 m 3 plus 2 m 4 plus m 5 is equal to 20 when n one and all m i greater than zero let us now make the following substitution say x 1 is equal to n 1 minus 1 x 2 is equal to m 2 minus 1 x 3 is equal to m 3 minus 1 x 4 is equal to m 4 minus 1 and x 5 is equal to m 5 minus 1.

therefore each x i is greater than equal to 0 because these we knew are positive now because we are subtracting 1 some of them may become 0 therefore 5 x 1 plus 4 x 2 plus 3 x 3 plus 2 x 4 plus x 5 is equal to 5 times n 1 minus 1 plus 4 times m 2 minus 1 plus 3 times m 3 minus 1 plus 2 times m 4 minus 1 plus m 5 minus 1 is equal to 20 minus 1 plus 2 plus 3 plus 4 plus 5 is equal to 20 minus 15 is equal to 5 therefore we get 5 x 1 plus 4 x two plus three x three plus two x four plus x five is equal to five when x i greater than equal to zero now five x one takes value zero 5 10 like that 4 x 2 takes values 0 4 8 twelve like that three x three takes values zero three six 2 x 2 takes values 0 2 4 6 and x sorry this is x 4 and x 5 takes values 0 1 2 3 like that therefore the power series for us is 1 plus x to the power 5 plus x to the power 10 multiplied by 1 plus x to the power 4 plus x to the power 8 multiplied by 1 plus x cube plus 6 6 multiplied by 1 plus 6 square plus 6 4 plus multiplied by 1 plus x plus x square plus 6 cube as you can understand this is corresponding to this set of values this is corresponding to this set of values and finally this is corresponding to this set of values we need to compute the coefficient of x to the power 5 since x to the power 5 is important we need not think anything of more than 5 in this any of these power series therefore the problem boils down to to compute coefficient of x to the power five in one plus x to the power five multiplied by one plus x to the power four multiplied by 1 plus x to the power 3 we are not going any further because next term is x to the power 6 1 plus x square plus x to the power 4 and this multiplied by 1 plus 6 plus 6 square plus x cube plus 6 to the power 4 plus x to the power 5 because we need all of them i need not go anything beyond that and we need to find out the coefficient of x to the power five

so let us start multiplication these two are giving me 1 plus x to the power 4 plus x to the power 5 these 2 are giving me 1 plus x square plus x to the power 4 plus x cube plus x to the power 5 plus x to the power 7 multiplied by 1 plus x plus x square plus x cube plus x to the power 4 plus x to the power 5 now since 7 is not needed we can ignore it therefore we have 1 plus x to the power 4 plus x to the power 5 multiplied by 1 plus x square plus x cube plus x to the power 4 plus x to the power 5 multiplied by one plus x plus x square plus x cube plus x to the power four plus x to the power five now let us multiply this we get 1 plus x square plus x cube plus x to the power 4 plus x to the power 5 by multiplying each one of them with 1 plus x to the power 4 plus x to the power 5 if i multiply these two with this one plus we see that all other terms are going to be greater than 5

so we do not need to get anything multiplied by 1 plus x plus up to x to the power 5 this is is equal to 1 plus 6 square plus 6 cube plus 2 x to the power 4 plus 2 x to the power 5 multiplied by 1 plus x plus up to x to the power 5 therefore coefficient of x to the power 5 is equal to 2 plus 2 plus 1 plus 1 plus 1 is equal to 7 therefore number of possible solutions is equal to seven have done a very similar problem 0 less than n 1 less than n 2 less than n 3 less than n 4 and we wanted that the total should be such that sigma ni i is equal to 1 to 4 is equal to 16 and we found that the number of solutions is equal to nine i leave that as an exercise you follow the same technique to find that the coefficient of x to the power 16 is coming out to be nine okay friends i stop here today

so in this six lectures we have solved many problems in probability i hope that these will help you in solving problems for your examinations where you may have to solve probability problems ok then thank you

so much you